

A UNIVERSAL A_∞ STRUCTURE ON BV ALGEBRAS WITH MULTIPLE ZETA VALUE COEFFICIENTS

JOHAN ALM

ABSTRACT. We construct an explicit and universal A -infinity deformation of Batalin-Vilkovisky algebras, with all coefficients expressed as rational sums of multiple zeta values. If the Batalin-Vilkovisky algebra that we start with is cyclic, then so is the A -infinity deformation. Moreover, the adjoint action of the odd Poisson bracket acts by derivations of the A -infinity structure. The construction conjecturally defines a new presentation of the Grothendieck-Teichmüller Lie algebra.

1. INTRODUCTION

A Batalin-Vilkovisky algebra is a differential graded commutative algebra \mathcal{A} together with a so-called BV operator $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ of degree -1 , such that the operation

$$\{f, g\} = \Delta(fg) - \Delta(f)g - (-1)^{|f|} f\Delta(g),$$

is a degree -1 graded Poisson bracket on \mathcal{A} , and such that the BV operator acts as derivations of this bracket. Batalin-Vilkovisky algebras were invented by physicists as a tool in the quantization of gauge theories, but there are several natural examples also in pure mathematics, such as Hochschild cohomology groups of Frobenius algebras, symplectic homology of symplectic manifolds, homology groups of free loop spaces and polyvector fields on manifolds with a specified volume form. The paper by Drummond-Cole and Vallette (2013) contains a survey of the subject, with detailed further references to the literature.

Every Batalin-Vilkovisky algebra is by definition a differential graded commutative algebra. In this paper we give a universal construction of an A_∞ deformation of this graded commutative product. Since the graded commutative product is already associative the higher homotopies of the A_∞ structure are philosophically analogous to Massey products, though this analogy is somewhat lacking, because the A_∞ structure is not homotopy commutative. Still, for a generically chosen Batalin-Vilkovisky algebra, the A_∞ deformation is not homotopic to the undeformed algebra.

The deformation that we construct is weakly canonical, in the following sense. Batalin-Vilkovisky algebras are algebras for the homology operad $H.(f\underline{M}_0)$ associated to the topological operad $f\underline{M}_0$ of Riemann spheres with marked punctures and phase parameters (angles) at each puncture, see Getzler (1994); Giansiracusa and Salvatore (2012). Analogously, associative algebras are governed by the homology operad $H.(X)$ of a topological operad X that parametrizes configurations of points on a line. Embedding the line as the real axis on a Riemann sphere (and fixing the phase parameters to point along that axis, say) defines a morphism of topological operads $X \rightarrow f\underline{M}_0$. The induced morphism

$$m : H.(X) \rightarrow H.(f\underline{M}_0)$$

is the morphism that trivially interprets the commutative product of a Batalin-Vilkovisky algebra as an associative product, and completely forgets the BV operator. Both operads X and $f\underline{M}_0$ are formal, meaning that there exists quasi-isomorphisms of differential graded operads

$$C.(X) \xleftarrow{\simeq} P \xrightarrow{\simeq} H.(X) \quad \text{and} \quad C.(f\underline{M}_0) \xleftarrow{\simeq} Q \xrightarrow{\simeq} H.(f\underline{M}_0)$$

connecting the respective differential graded operad of (singular) chains to the respective homology operad. It is natural to ask if the natural morphism $X \rightarrow f\underline{M}_0$ is formal too, in the sense that the zig-zags of

quasi-isomorphisms can be chosen in such a way that we obtain a commutative diagram

$$\begin{array}{ccc}
C.(X) & \longrightarrow & C.(f\underline{M}_0) \\
\uparrow \simeq & & \uparrow \simeq \\
Q & \longrightarrow & P \\
\downarrow \simeq & & \downarrow \simeq \\
H.(X) & \xrightarrow{m} & H.(f\underline{M}_0).
\end{array}$$

It follows from recent results of Tourtchine and Willwacher (2014) that we can *not*. One may interpret this as saying that the map $m : H.(X) \rightarrow H.(f\underline{M}_0)$ is much less canonical than we would like to think it is. Rather, it is a mere shadow or first order approximation of the “truly canonical” topological morphism $X \rightarrow f\underline{M}_0$. The universal A_∞ structure that we construct adds the necessary higher order correction terms to the map m , and is in this sense canonical. However, note that the zig-zags of quasi-isomorphisms (as above) are themselves very much not canonical, so the canonical nature of our structure is rather weak. There is essentially a unique zig-zag witnessing the formality of the operad X . Formalities of the operad $f\underline{M}_0$, on the other hand, are by Ševera (2010) essentially parametrized by the set of Drinfeld associators. We conjecture that the universal A_∞ structure that we construct corresponds to choosing the Knizhnik-Zamolodchikov associator, which in a sense is the canonical choice of associator once we strictify the setup from purely topological to one of algebraic geometry, since then the Knizhnik-Zamolodchikov arises canonically from the comparison isomorphism between Betti and de Rham realizations.

Assume k_1, \dots, k_r is a sequence of strictly positive integers, with $k_r \geq 2$. The multiple zeta values (for short, MZVs) are the real numbers of the form

$$\zeta(k_1, \dots, k_r) = \sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}}.$$

It is relatively easy, using the above displayed series representation, to see that the multiplication of two MZVs is a rational linear combination of MZVs, so they span a subalgebra ζ of the real numbers. The Knizhnik-Zamolodchikov associator is known to be a kind of generating function for multiple zeta values. Similarly, all the coefficients of our A_∞ structure are rational sums of MZVs, and every MZV potentially contributes. In more detail, we prove that the n -th higher homotopy of our structure has coefficients given by rational sums of only MZVs that have $k_1 + \dots + k_r \leq n - 3$. It follows that our construction defines a morphism

$$\varphi_\nu : H^1(\text{Def}(\text{As}_\infty \xrightarrow{m} H.(f\underline{M}_0)))' \rightarrow \zeta^+ / \zeta^+ \cdot \zeta^+,$$

from the (finite, graded) dual of the degree one cohomology group of the deformation complex of the map $m : H.(X) \rightarrow H.(f\underline{M}_0)$, to the quotient of the augmentation ideal of the MZV algebra by all nontrivial products. This is analogous to the mapping

$$\varphi_{\text{KZ}} : \text{grt}'_1 \oplus \mathbb{Q}\zeta(2) \rightarrow \zeta^+ / \zeta^+ \cdot \zeta^+$$

defined by the Knizhnik-Zamolodchikov associator. The map φ_{KZ} is famously conjectured to be an isomorphism, and one may conjecture that our φ_ν is likewise an isomorphism.

Briefly, our construction proceeds as follows. Let $M_{0,n}^\delta$ be Francis Brown’s partial compactification of the moduli space of genus zero curves with n marked points (Brown (2009)). It sits as an intermediary $M_{0,n}^\delta \subset M_{0,n}^\delta \subset \overline{M}_{0,n}$ between the open moduli space and the Deligne-Mumford compactification, as the space obtained by adding only those boundary divisors of the Deligne-Mumford compactification that bound the connected component of the set of real points $M_{0,n}(\mathbb{R})$ which corresponds to having the marked points in the canonical order $z_1 < \dots < z_n$. The closure of this connected component inside $M_{0,n}^\delta(\mathbb{C})$, call it X_n , is an associahedron of dimension $n - 3$. Brown’s moduli spaces constitute a planar operad (we may cyclically permute the marked points, but general permutations are not allowed). The associahedra X_n likewise form a planar operad, and using results from Brown’s thesis (Brown (2009)) there are natural maps

$$H^*(M_{0,n}^\delta) \rightarrow \Omega_{\text{dR}}^*(X_n),$$

by which the rational cohomology cooperad of the moduli spaces can be represented as certain logarithmic differential forms on associahedra. The main result of Brown (2009) is that every top-dimensional form in its image has an integral over the associahedron that converges to a rational sum of MZVs. Thus, there exists a canonical morphism of planar operads

$$\zeta \otimes_{\mathbb{Q}} \text{As}_{\infty} \rightarrow \zeta \otimes_{\mathbb{Q}} H.(M_0^{\delta}),$$

defined by integration pairing. Let \mathfrak{d}_n be Brown's Lie algebra of dihedral braids (Brown (2009)). It has a generator $\delta_{ij} = \delta_{ji}$ for each pair of indices $1 \leq i \neq j \leq n$ with $i \neq j+1$ modulo n , and relations

$$[\delta_{i-1j} + \delta_{ij-1} - \delta_{i-1j-1} - \delta_{ij}, \delta_{k-1l} + \delta_{kl-1} - \delta_{k-1l-1} - \delta_{kl}] = 0$$

for all quadruples of distinct indices. We note that this family of Lie algebras form a planar operad. Brown gave a novel treatment of the Knizhnik-Zamolodchikov connection in terms of this Lie algebra, and based on his results we show how to write down a morphism of differential graded cooperads

$$\alpha_{\text{reg}} : C^*(\mathfrak{d}) \rightarrow H^*(M_0^{\delta}).$$

This map can be regarded as a “regularized” version of the Knizhnik-Zamolodchikov connection, but note that it is not a morphism of cooperads of differential graded commutative algebras, only of cooperads of differential graded vector spaces, hence it is not a connection in the true sense of being defined by a connection form with values in the Lie algebra of dihedral braids.

Let \mathfrak{rb}_n be the Lie algebra of spherical ribbon braids. We use a slightly nonstandard presentation of this Lie algebra, with generators b_{ij} ($1 \leq i \neq j \leq n$) and s_k ($1 \leq k \leq n$), and relations saying that $b_{ji} = b_{ij}$, that $[b_{ij}, b_{kl}] = 0$ whenever all four indices are distinct, that $2s_k + \sum_{i=1}^n b_{ik} = 0$ for all k , and that all the s_k are central. These Lie algebras assemble, for varying n , to an operad. Its associated differential graded operad $H.(\mathfrak{rb})$ of Chevalley-Eilenberg homologies may be identified with the operad of Batalin-Vilkovisky algebras, which we previously in the introduction denoted $H.(f\mathcal{M}_0)$. We construct a morphism

$$\gamma : C.(\mathfrak{d}) \rightarrow C.(\mathfrak{rb})$$

of differential graded planar operads of Chevalley-Eilenberg chain complexes. By suitably dualizing, taking homology, and composing with our regularized Knizhnik-Zamolodchikov connection, we obtain a morphism

$$\gamma \circ \alpha_{\text{reg}}^* : H.(M_0^{\delta}) \rightarrow H.(\mathfrak{rb}).$$

Composing with the morphism defined by Brown's integration pairing then defines a morphism

$$\nu : \zeta \otimes_{\mathbb{Q}} \text{As}_{\infty} \rightarrow \zeta \otimes_{\mathbb{Q}} H.(\mathfrak{rb})$$

of planar differential graded operads.

In the last section of the paper we prove that the adjoint action of the odd Poisson bracket of a Batalin-Vilkovisky algebra acts by strict derivations of our A_{∞} structure. To prove this we define a family of manifolds with corners $X_{p,q} \subset f\mathcal{M}_{0,p+q}$ that encode the two-colored operadic combinatorics of the homotopies of an L_{∞} action by A_{∞} derivations of an A_{∞} algebra. Our construction, based on integration over the associahedra X_n , extends to a representation by integration over also the spaces $X_{p,q}$, and inspection of the involved integrals shows that the data added by this is just the statement that the adjoint action of the odd Poisson bracket is an action by derivations of the A_{∞} structure. In formulas, the adjoint action is a morphism of differential graded Lie algebras

$$\text{ad} : (\mathcal{A}[1], \{ , \}) \rightarrow \text{Der}(\mathcal{A}, \nu).$$

It follows that if κ is a Maurer-Cartan element of $\mathcal{A}[1]$, then ν will be an A_{∞} structure also on \mathcal{A} with the differential twisted by addition of the term ad_{κ} .

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2. PRELIMINARY DEFINITIONS

The set $\{1, \dots, n\}$ is denoted $[n]$. We usually treat it as (cyclically) ordered in the obvious way. The cardinality of a set S is denoted $\#S$.

Let \mathbb{K} be a field of characteristic zero. The terms *differential graded vector space*, henceforth abbreviated to *dg vector space*, and *cochain complex* (over \mathbb{K}) are used as synonyms. In particular, we always use cohomological grading so that differentials increase degree. The *degree* of a homogeneous $u \in V^d$ is written $|u| = d$. We always apply the Koszul sign rules for tensor products of dg vector spaces and tensor products of maps of dg vector spaces. Briefly, these sign rules are as follows.

The space of maps from V to W is the dg vector space $\text{Map}(V, W)$ with

$$\text{Map}(V, W)^n = \prod_p \text{Hom}_{\mathbb{K}}(V^{p-n}, W^p),$$

where $\text{Hom}_{\mathbb{K}}(V^{p-n}, W^p)$ denotes the vector space of all linear maps from V^{p-n} to W^p , and differential given on $\phi \in \text{Map}(V, W)^n$ by $d_{\text{Map}(V, W)}^n \phi = d_W \circ \phi - (-1)^n d_V \circ \phi$. A vector ϕ of $\text{Map}(V, W)^n$ is called a map of dg vector spaces of degree n . Note that a morphism from V to W is the same thing as a cocycle of degree 0 of $\text{Map}(V, W)$. We apply the Koszul sign rules to maps, which says that for homogeneous maps f, g and homogeneous vectors u, v in their respective domains, $f \otimes g$ is defined by $(f \otimes g)(u \otimes v) = (-1)^{|g||u|} f(u) \otimes g(v)$. Given dg vector spaces V and W their tensor product is the dg vector space $V \otimes W$ with $(V \otimes W)^n = \bigoplus_{p+q=n} V^p \otimes_{\mathbb{K}} W^q$, differential defined by $d_{V \otimes W} = d_V \otimes id_W + id_V \otimes d_W$ (using the Koszul sign rule for maps). The Koszul symmetry for $V \otimes W$ is the morphism

$$\sigma_{V \otimes W} : V \otimes W \rightarrow W \otimes V$$

given on vectors of homogeneous degree by $\sigma_{V \otimes W}(v \otimes w) = (-1)^{|v||w|} w \otimes v$. The tensor product, the Koszul symmetry and the tensor unit \mathbb{K} give the category of dg vector spaces the structure of a symmetric monoidal category. Using the space of maps and the Koszul sign rules for maps we can (and implicitly usually will do) consider the category of dg vector spaces as a category enriched in itself, because the space of maps and the tensor product satisfy the usual adjunction.

2.1. Planar operads. All dg (co)operads are assumed to be (co)augmented, and we will accordingly dispense with the distinction between dg (co)operads and dg pseudo-(co)operads. We otherwise follow the conventions concerning operads adopted in Loday and Vallette (2012). A notable exception is the terminology of *planar* (co)operads, or what one might also term *nonsymmetric cyclic (co)operad*. These feature in Menichi (2004); Alm and Petersen (2015); Dupont and Vallette (2015). The idea for the concept is very simple: just like cyclic operads are based on trees, operads on rooted trees, and nonsymmetric operads on planar rooted trees, planar operads are based on planar (non-rooted) trees.

A *graph* G is a finite set of flags F_G with an involution $\tau : F_G \rightarrow F_G$, a finite set of vertices V_G and a function $h : F_G \rightarrow V_G$. The fixed points of τ are called legs and the orbits of length two are called edges. Let E_G denote the set of edges. Let v and v' be two vertices. They are said to share an edge if there exists a flag f such that $h(f) = v$ and $h(\tau(f)) = v'$, and they are said to be connected if there exists a sequence of vertices $v = v_0, v_1, \dots, v_k = v'$ such that v_i and v_{i+1} share an edge. A graph is called connected if any two of its vertices are connected. The valency of a vertex is the cardinality $\#h^{-1}(v)$. A morphism of graphs $\phi : G \rightarrow G'$ is a function $\phi^* : F_{G'} \rightarrow F_G$, which is required to be bijective on legs and injective on edges, together with a function $\phi_* : V_G \rightarrow V_{G'}$, such that ϕ_* is a coequalizer of the two functions $h, h \circ \tau : F_G \setminus \phi^*(F_{G'}) \rightarrow V_G$.

A graph is called a *tree* if it is connected and $\#V_G - \#E_G = 1$. A tree is *planar* if for each vertex v there is a specified cyclic ordering on the set $F_v = h^{-1}(v)$ of flags attached to v . A tree is *stable* if every vertex has at least three legs attached to it. A tree (with n legs) is said to be *labeled* if we are given a bijection between the set of legs and the cyclically ordered set $[n]$.

A *morphism* of stable, planar and labeled trees is a morphism of the underlying graphs that respects all cyclic orderings on flags and legs. With these conventions stable, planar labeled trees form a category PT .

Definition 2.1.1. Fix a cocomplete monoidal category \mathbf{V} , such that $-\otimes-$ is cocontinuous in both variables. A *planar collection* in \mathbf{V} is an indexed family $\{K_n \mid n \geq 3\}$ of objects in \mathbf{V} , such that K_n is a representation of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Such collections form a category $\text{PT}(\mathbf{V})$. Moreover, every planar collection K

defines a functor

$$K[\] : \text{Iso PT} \rightarrow \mathbf{V}$$

on the category of stable, planar, labeled trees and their isomorphisms, via

$$K[T] = \bigotimes_{v \in V_T} K_{n(v)}.$$

Above $n(v)$ is the number of half-edges adjacent to the vertex. To be precise, instead of $K_{n(v)}$ one should write

$$\left(\bigoplus_{F_v \cong \{1, \dots, n\}} K_{n(v)} \right)_{\mathbb{Z}/n\mathbb{Z}},$$

a sum over order-preserving bijections between F_v – the cyclically ordered set of half-edges adjacent to v – and a standard cyclically ordered set. This can be used to define an endofunctor $\text{TP}^{\text{pl}} : \text{PT}(\mathbf{V}) \rightarrow \text{PT}(\mathbf{V})$ by

$$\text{TP}^{\text{pl}}(K)_n = \text{colim}(\text{Iso}(\text{PT} \downarrow t_n) \xrightarrow{K[\]} \mathbf{V}).$$

In the above t_n is a labeled planar tree with a single vertex and n legs, and $(\text{PT} \downarrow t_n)$ denotes the comma category of trees over t_n . We call TP^{pl} the *free planar operad functor*.

Assume that T is a stable, planar labeled tree and that for every vertex $u \in V_T$ of T we are given a stable planar labeled tree T_u . Then we can build a tree T' that contains each T_u as a subtree and has the property that contracting all the T_u subtrees of T' produces the original tree T . In particular, $V_{T'} = \bigsqcup_{u \in V_T} V_{T_u}$, giving a canonical morphism

$$\bigotimes_{u \in V_T} \bigotimes_{v \in V_{T_u}} K_{n(v)} \rightarrow \bigotimes_{w \in V_{T'}} K_{n(w)}.$$

These maps assemble to a natural transformation $\text{TP}^{\text{pl}} \circ \text{TP}^{\text{pl}} \rightarrow \text{TP}^{\text{pl}}$. The definition as a colimit gives a natural transformation $\text{id} \rightarrow \text{TP}^{\text{pl}}$. Together these two natural transformations give the free planar operad functor the structure of a monad.

Definition 2.1.2. A *planar (pseudo-)operad* in \mathbf{V} is an algebra for the free planar operad monad. A *morphism* of planar (pseudo-)operads is a morphism of algebras for the free planar operad monad.

Given the established terminology among operadchiks, planar operads should perhaps more properly be called “nonsymmetric cyclic” operads, but to the author’s ears that sounds a bit forced.

Remark 2.1.3. Just as for ordinary cyclic operads, a planar operad is determined by a planar collection \mathbf{O} and a family of *composition* morphisms

$$\circ_i^j : \mathbf{O}_n \otimes \mathbf{O}_k \rightarrow \mathbf{O}_{n+k-2},$$

parametrized by $1 \leq i \leq n$, $1 \leq j \leq k$, satisfying certain associativity and equivariance conditions. These morphisms arise as follows. Graft the i th vertex of the tree t_n to the j th vertex of t_k , to obtain a tree $t_n \circ_i^j t_k$: the composition of \mathbf{O} is the morphism

$$\mathbf{O}[t_n \circ_i^j t_k] \rightarrow \mathbf{O}[t_{n+k-2}]$$

defined by the algebra structure $\text{TP}^{\text{pl}}(\mathbf{O}) \rightarrow \mathbf{O}$. In fact, only the operations

$$\circ_i^{n+1} : \mathbf{O}_{m+1} \otimes \mathbf{O}_{n+1} \rightarrow \mathbf{O}_{m+n}, \quad 1 \leq i \leq n,$$

suffice. We could have defined a planar (pseudo-)operad as a stable collection \mathbf{O} such that the collection $\{\mathbf{O}(n) = \mathbf{O}_{n+1}\}_{n \geq 2}$ together with the operations $\circ_i = \circ_i^{n+1}$ is a nonsymmetric (pseudo-)operad and, moreover, if $\tau : \mathbf{O}(n) \rightarrow \mathbf{O}(n)$ is the right action of the cycle $(n+1 \ 1 \ \dots \ n)$, then

$$(\phi \circ_1 \psi)\tau = \psi\tau \circ_n \phi\tau, \quad \forall \phi \in \mathbf{O}(m), \psi \in \mathbf{O}(n), m, n \geq 2,$$

while

$$(\phi \circ_i \psi)\tau = \phi\tau \circ_{i-1} \psi\tau, \quad \forall \phi \in \mathbf{O}(m), \psi \in \mathbf{O}(n), m, n \geq 2, 2 \leq i \leq m.$$

These are the axioms we typically verify, but the freedom to graft arbitrary planar trees simplifies many abstract arguments.

Let us specialize now to the case when \mathbf{V} is the category of dg vector spaces, with Koszul sign rules. We can then define a slight variation of the free planar operad functor, as follows. Define

$$\det \otimes K[] : T \mapsto \det(V_T) \otimes_R K[T],$$

where the determinant $\det(S)$ of a finite set S is defined to be the top exterior power $\wedge^{\#S} \mathbb{K}^S$, placed in degree zero. The formula

$$\mathrm{T}^{\mathrm{pl},-}(K)_n = \mathrm{colim}(\mathrm{Iso}(\mathrm{PT} \downarrow t_n) \xrightarrow{\det \otimes K[]} \mathbf{V})$$

again defines a monad.

Definition 2.1.4. The *free antiplanar monad* is the functor $\mathrm{T}^{\mathrm{pl},-}$. The algebras of this monad are called *antiplanar dg (pseudo-)operads*.

Remark 2.1.5. By suitably dualizing the definitions we obtain notions of *(anti)planar (pseudo-)cooperads*. The linear dual of an (anti)planar dg (pseudo-)cooperad is an (anti)planar dg (pseudo-)operad.

The assumption that \mathbf{V} is the category of dg vector spaces implies that the functors T^{pl} and $\mathrm{T}^{\mathrm{pl},-}$ are not only monads, but also in a natural way *comonads*. The structure map

$$\mathrm{T}^{\mathrm{pl}} \rightarrow \mathrm{T}^{\mathrm{pl}} \circ \mathrm{T}^{\mathrm{pl}}$$

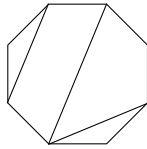
is given by “decomposing trees”. The counit is given by projection onto trees with a single vertex. Coalgebras for the comonad T^{pl} are *conilpotent planar cooperads*, and coalgebras for $\mathrm{T}^{\mathrm{pl},-}$ are *conilpotent antiplanar cooperads*.

Convention 2.1.6. All cooperads in this paper will be conilpotent. A *cofree* (anti)planar cooperad will, hence, refer to to a cooperad of the form $\mathrm{T}^{\mathrm{pl}}(K)$ (resp. $\mathrm{T}^{\mathrm{pl},-}(M)$) for some planar collection K .

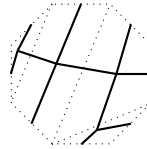
Remark 2.1.7. Every (anti)cyclic operad defines a (anti)planar operad, by simply forgetting the information that we are allowed to arbitrarily permute the inputs, retaining only the information that we are allowed to cyclically permute them. Accordingly, the operad Lie of Lie algebras, for example, is a planar operad. The operad of associative algebras \mathbf{Ass} is likewise cyclic. Moreover, as a symmetric operad it is freely generated by a nonsymmetric operad \mathbf{As} . The same is true once we add the freedom to cyclically permute inputs: the cyclic operad $\mathbf{Ass}_n = \mathrm{Ind}_{\mathbb{Z}/n\mathbb{Z}}^{\Sigma_n} \mathbb{Q}$ is generated by the planar (pseudo-)operad $\mathbf{As}_n = \mathbb{Q}$.

Convention 2.1.8. We will in the main body of the paper drop the qualifying prefix “pseudo” in front of operads and cooperads.

2.2. Using polygons instead of trees. To every stable planar tree with n legs one can associate a tessellation of the oriented standard n -gon, as shown in the figures below:



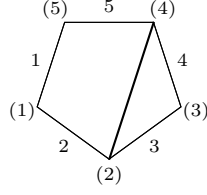
A tessellation of an oriented octagon.



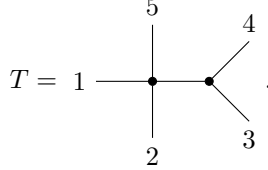
The planar tree dual to the tessellation.

We make extensive use of this dual language in this paper, so let us develop it slightly more detail. For every cyclically ordered set S of cardinality at least three, let $\chi_1(S)$ denote the set of unordered pairs $\{i, j\}$ of indices $i, j \in S$ that are not consecutive in the cyclic order. We identify $\chi_1(S)$ as the set of *chords* on the $\#S$ -gon with sides labeled by S . If $S = [n]$, then we write $\chi_1(S) = \chi_1(n)$. We consider the $\#S$ -gon as oriented by embedding it in the plane in such a way that the cyclic order of S concides with the counterclockwise

ordering define by the plane. We number the vertices so the side i is oriented from vertex $i - 1$ to vertex i . Below is the standard pentagon, with the chord $\{2, 4\}$ drawn on it:



This is dual to the tree



Define a *tessellation* of the S -gon to be a collection $T = \{c_1, \dots, c_r\}$ of some number r of non-crossing chords $c_i \in \chi_1(S)$. Then T is equivalent to a planar tree with $r - 1$ vertices, and set of legs S . Define $\chi_r(S)$ to be the set of tessellations by r chords, and let $\chi_0(S) = S$. Then the free planar operad on a planar collection K can equivalently be described by a formula

$$\mathrm{Tp}^{\mathrm{pl}}(K)_n = \bigoplus_{r \geq 0} \bigoplus_{T \in \chi_r(n)} K[T],$$

where, somewhat informally,

$$K[T] = \bigotimes_{S_i} K_{S_i}$$

is a tensor over the subpolygons S_i defined by the tessellation T . (To be precise we define $K[T]$ as $K[T]$, the latter defined by the planar tree dual to the tessellation.) In this dual language operadic compositions corresponds to gluing polygons, and cooperadic cocomposition corresponds to splitting polygons along chords.

2.3. Deformation theory of planar operads. Consider the category of (reduced and augmented, symmetric) dg operads. This category has a model structure induced by that on dg vector spaces. A morphism $f : \mathbf{P} \rightarrow \mathbf{P}'$ is

- * a weak equivalence, also referred to as a quasi-isomorphism, if each $f_n : \mathbf{P}(n) \rightarrow \mathbf{P}'(n)$ is a quasi-isomorphism of dg vector spaces.
- * a fibration if each f_n is a fibration of dg vector spaces.
- * a cofibration if it satisfies the lifting property.

For a proof, see Hinich (1997).

Claim 2.3.1. The definition of a model structure repeats mutatis mutandum to define a model structure on the category of planar dg (pseudo-)operads.

The free operad functor defines a bar-cobar duality and cofibrant replacements. A consequence of the above claim is that the free planar operad functor defines a bar-cobar duality and cofibrant replacements for planar dg operads. In other words, all the theory concerning deformation complexes and Koszul duality of Loday and Vallette (2012); Merkulov and Vallette (2009); Getzler and Kapranov (1995) applies to planar dg (co)operads, too.

2.4. Analytical manifolds with corners. We here record some facts from Brown (2009), and Alekseev et al. (2012).

Set $U_{p,q} = \mathbb{R}_{\geq 0}^q \times \mathbb{R}^p$, $d = p + q$. Define a non-permuting analytic isomorphism $\varphi : U_{p,q} \rightarrow U_{p,q}$ to be an analytic diffeomorphism $\varphi = (\varphi_1, \dots, \varphi_d)$ of \mathbb{R}^d with positive Jacobian, restricting to a diffeomorphism of $U_{p,q}$, where ϕ_i satisfies $\phi_i|_{x_i=0} = 0$ and $(\partial \phi_i / \partial x_i)|_{x_i=0} = 1$ for each $i = 1, \dots, q$. There is also a natural action of the permutation group $\Sigma_p \times \Sigma_q$ on $U_{p,q}$, and we define an analytic isomorphism $U_{p,q} \rightarrow U_{p,q}$ to be a map as above composed with such a permutation. An analytic manifold with corners is a manifold with

an atlas of that has analytic isomorphisms $U_{p,q} \rightarrow U_{p,q}$ as transitions. Denote the algebra of complex-valued real-analytic functions on $U_{p,q}$ by $C^\omega(U_{p,q})$. Set

$$\begin{aligned}\Omega(U_{p,q}) &= C^\omega(U_{p,q})[dx_1, \dots, dx_{p+q}], \\ \Omega_1^\bullet(U_{p,q}) &= \Omega(U_{p,q})[d \log x_1, \dots, d \log x_q], \\ \Omega_{p,\log}^\bullet(U_{p,q}) &= \Omega(U_{p,q})[\log x_1, \dots, \log x_q, d \log x_1, \dots, d \log x_q].\end{aligned}$$

One may check that these graded vector spaces are invariantly defined under analytic isomorphisms of $U_{p,q}$; hence can be defined for any analytic manifold with corners. Moreover, the de Rham differential extends to give $\Omega_{p,\log}(U_{p,q})$ the structure of a dg vector space.

Let $l \in [q]$ and let $D = x_l = 0 \subset U_{p,q}$. The regularized restriction along D is the linear map

$$\text{Reg}_D : \Omega_{p,\log}^\bullet(U_{p,q}) \rightarrow \Omega_{p,\log}^\bullet(D)$$

given by

$$\text{Reg}_D(\alpha) = \alpha(x_l = \log x_l = dx_l = d \log x_l = 0).$$

This is compatible with analytic isomorphisms and the de Rham differential; hence defines regularized restrictions along codimension one boundary strata of any analytic manifold with corners. It also restricts functorially to the subsheaves $\Omega^\bullet \subset \Omega_1^\bullet \subset \Omega_{p,\log}^\bullet$; on Ω^\bullet it is ordinary restriction of forms.

3. BROWN'S MODULI SPACES

This section recollects facts concerning Brown's moduli spaces $M_{0,n}^\delta$, mostly borrowing from Brown (2009). Define the open moduli space of n -pointed genus zero curves as the quotient manifold

$$M_{0,n} = ((\mathbb{CP}^1)^n \setminus \text{diagonals}) / \text{PGL}_2(\mathbb{C}).$$

It is an algebraic variety and the ring of functions has the following presentation. Define $\chi_1(n)$ to be the set of unordered pairs $\{i, j\}$ of indices $i, j \in [n]$ that are not consecutive modulo n . We shall follow Brown and refer to $\chi_1(n)$ as the set of *chords* on $[n]$. Given a chord $\{i, j\} \in \chi_1(n)$, let u_{ij} denote the cross-ratio

$$u_{ij} = [i \ i + 1 \mid j + 1 \ j] = \frac{(z_i - z_{j+1})(z_{i+1} - z_j)}{(z_i - z_j)(z_{i+1} - z_{j+1})}.$$

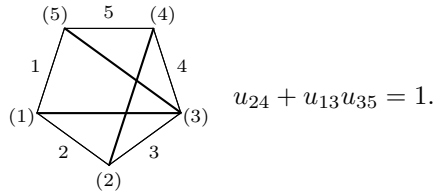
It is well-defined as a function on $M_{0,n}$. Considering $[n]$ as cyclically ordered in the natural way, any chord $\{i, j\}$ will partition $[n] \setminus \{i, j\}$ into two connected components. Say that two chords $\{i, j\}$ and $\{k, l\}$ *cross* if k and l belong two different connected components in the partition defined by $\{i, j\}$. (This is obviously a symmetric condition in the sense that this is true if and only if i and j lie in different connected components of the partition defined by $\{k, l\}$.) Given a subset $A \subset \chi_1(n)$, let A^\perp denote the set of chords that cross every chord in A , and say that two subsets $A, B \subset \chi_1(n)$ *cross completely* if $A^\perp = B$ and $B^\perp = A$. The collection of arbitrary cross-ratios, $\{[i \ j \mid k \ l]\}$, is well-known to generate the ring of functions on the moduli space. Following Brown, one can argue based on the various symmetries satisfied by cross-ratios, that the functions u_{ij} are in fact sufficient to generate the whole ring of functions:

$$\mathcal{O}(M_{0,n}) = \mathbb{Q}[u_{ij}, u_{ij}^{-1} \mid \{i, j\} \in \chi_1(n)] / \langle R \rangle,$$

where R is spanned by all elements

$$1 - \prod_{\{i,j\} \in A} u_{ij} - \prod_{\{k,l\} \in B} u_{kl},$$

labeled by pairs of completely crossing subsets $A, B \subset \chi_1(n)$. We shall follow Brown and refer to the u_{ij} 's as the *dihedral coordinates*. The image below shows crossing chords on a pentagon and the corresponding relation between coordinate functions on $M_{0,5}$.



Definition 3.0.1. Define A_n to be the de Rham complex of logarithmic algebraic forms on $M_{0,n}$ with possible singularities on $\overline{M}_{0,n} \setminus M_{0,n}$.

It follows from Deligne (1974) that A_n has trivial differential and that $A_n \rightarrow H^\bullet(M_{0,n})$ is injective. The cohomology of the spaces $M_{0,n}$ have a pure Hodge structure and that implies that the map into the cohomology is also surjective.

Lemma 3.0.2. (Brown (2009)) The cohomology algebra $A_n \cong H^\bullet(M_{0,n})$ is the graded commutative algebra generated by the degree 1 elements

$$\alpha_{ij} = d \log u_{ij}, \quad \{i, j\} \in \chi_1(n),$$

modulo the relations that

$$\left(\sum_{\{i,j\} \in A} \alpha_{ij} \right) \left(\sum_{\{k,l\} \in B} \alpha_{kl} \right) = 0$$

for all pairs of completely crossing subsets $A, B \subset \chi_1(n)$.

Note that the logarithmic forms α_{ij} satisfy the relations given above *as forms*, not just as cohomology classes.

Definition 3.0.3. Brown's moduli space $M_{0,n}^\delta$ is the variety

$$M_{0,n}^\delta = \text{Spec } \mathbb{Q}[u_{ij} \mid \{i, j\} \in \chi_1(n)] / \langle R \rangle,$$

where R is the same set of relations as that defining the open moduli space.

3.1. Operadic structure. Choose a chord on $[n]$. Without loss of generality we may write the chord in the form $\{i, i+k\}$, with $1 \leq i$ and $i+k \leq n$. The chord $\{i, i+k\}$ divides the n -gon into two: a $(k+1)$ -gon with vertices $\{1, \dots, i, i+k, \dots, n\}$ and an $(n-k+1)$ -gon with vertices $\{i, i+1, \dots, i+k\}$. This defines a partition

$$\chi_1(n) = \chi_1(\{1, \dots, i, i+k, \dots, n\}) \sqcup \chi_1(\{i, i+1, \dots, i+k\}) \sqcup \{\{i, j\}\} \sqcup \{\{p, q\} \in \chi_1(n) \mid \{p, q\} \text{ crosses } \{i, j\}\}.$$

Define a morphism

$$\begin{aligned} \tilde{\Delta}_{\{i, i+k\}} : \mathbb{Q}[u_{ij} \mid \{i, j\} \in \chi_1(n)] \\ \rightarrow \mathbb{Q}[u_{ij} \mid \{i, j\} \in \chi_1(\{1, \dots, i, i+k, \dots, n\})] \otimes \mathbb{Q}[u_{ij} \mid \{i, j\} \in \chi_1(\{i, i+1, \dots, i+k\})] \end{aligned}$$

by sending u_{rs} to

$$\begin{aligned} u_{rs} \otimes 1, & \text{ if } \{r, s\} \in \chi_1(\{1, \dots, i, i+k, \dots, n\}), \\ 1 \otimes u_{rs}, & \text{ if } \{r, s\} \in \chi_1(\{i, i+1, \dots, i+k\}), \\ 0, & \text{ if } \{r, s\} = \{i, i+k\}, \\ 1 \otimes 1, & \text{ if } \{r, s\} \text{ crosses } \{i, i+k\}. \end{aligned}$$

By reindexing according to the unique order-preserving bijections

$$\{1, \dots, i, i+k, \dots, n\} \cong \{1, \dots, n-k+1\}, \quad \{i, i+1, \dots, i+k\} \cong \{1, \dots, k+1\},$$

we obtain a morphism

$$\Delta_{\{i, i+k\}} : \mathcal{O}(M_{0,n}^\delta) \rightarrow \mathcal{O}(M_{0,n-k+1}^\delta) \otimes \mathcal{O}(M_{0,k+1}^\delta),$$

and hence, dually, a morphism

$$\circ_i^{k+1} : M_{0,n-k+1}^\delta \times M_{0,k+1}^\delta \rightarrow M_{0,n}^\delta.$$

Lemma 3.1.1. (Brown (2009)) The morphism displayed above is the inclusion of the boundary strata of $M_{0,n}^\delta$ which is defined by the equation $u_{i, i+k} = 0$.

Lemma 3.1.2. The inclusions of boundary strata define a structure of planar operad (in the category of affine varieties) on the collection of Brown's moduli spaces.

Proof. The Deligne-Mumford compactifications $\overline{M}_{0,n}$ are well-known to assemble to a cyclic operad, with composition maps defined by inclusions of boundary strata. The necessary associativity and equivariance relations for the maps on Brown's moduli spaces follow immediately from the corresponding ones for the Deligne-Mumford compactifications. \square

3.2. The associahedra. The condition that the n marked points appear in the cyclic order $z_1 < \dots < z_n$ defines a connected component $X_n^o \subset M_{0,n}(\mathbb{R})$ of the set of real points of the open moduli space. Its compactification inside $M_{0,n}^\delta(\mathbb{R})$, call it X_n , is isomorphic as an analytical manifold with corners to the $(n-3)$ -dimensional associahedron. It is defined by the equations

$$X_n = \{0 \leq u_{ij} \leq 1 \text{ for all } \{i, j\} \in \chi_1(n)\},$$

and its interior is isomorphic to the open simplex $\{0 < z_2 < \dots < z_{n-2} < 1\}$. The planar operad structure of Brown's moduli spaces restricts to a planar operad structure on the X_n 's (in the category of oriented analytical manifolds with corners). The homology operad $H_*(X)$ is canonically isomorphic to the planar operad \mathbf{As} governing cyclic associative algebras. It is generated by a degree zero $\nu \in \mathbf{As}_3$ satisfying the relations $\nu \cdot (312) = \nu$ and $\nu \circ_1^3 \nu = \nu \circ_2^3 \nu$. The boundary of X_n is stratified by products of lower-dimensional X_l 's. It follows that the face complexes $C_*(X_n)$ constitute a planar dg suboperad of the operad of chains (currents) on X . One easily sees that $C_*(X)$ is free as a planar graded operad on the collection $\{[X_n] \mid n \geq 3\}$ of fundamental chains.

Lemma 3.2.1. The identification $\nu_n = [X_n]$ is an isomorphism between the planar dg operad $C_*(X)$ of fundamental chains on Brown's associahedra and the nonsymmetric dg operad \mathbf{As}_∞ of A_∞ algebras, considered as a planar dg operad.

Remark 3.2.2. We leave the lemma without proof. The isomorphism between the nonsymmetric A_∞ operad and the cell complex operad of associahedra, and the cyclic compatibility of it, is well-known. (The statement is old enough to predate the language of operads, and has historically been one of the main reasons for inventing the language of operads!)

3.3. Cooperads of cohomology algebras. One can regard the cohomology algebras A_n of the open moduli spaces as (possibly) singular analytical forms on Brown's moduli spaces, or even on the embedded associahedra, i.e., consider $A_n \rightarrow \Omega_1^*(X_n)$, cf. 2.4. One may accordingly apply regularized restriction of these forms to boundary strata.

Lemma 3.3.1. The collection \mathbf{A} of cohomology algebras of the open moduli spaces is a planar dg cooperad under the cocompositions defined by regularized restriction to boundary strata. If $\{k, l\} \in \chi_1(n)$ equals $\{i, j\}$ or crosses it, then the cocomposition

$$\Delta_{\{i,j\}}^{\mathbf{A}} : A_n \rightarrow A_{n-n'+1} \otimes A_{n'+1}$$

corresponding to regularized restriction of forms to the strata $u_{ij} = 0$ sends the form α_{kl} to zero; otherwise it just sends it to the form associated to the corresponding chord on $[n - n' + 1]$ or on $[n' + 1]$.

Proof. This is just the definition of regularized restriction, given the explicit form of the cocomposition $\mathcal{O}(M_{0,n}^\delta) \rightarrow \mathcal{O}(M_{0,n-n'+1}^\delta) \otimes \mathcal{O}(M_{0,n'}^\delta)$ on functions, since $u_{ij} = 0$ is a global equation defining the strata. \square

Ezra Getzler defined a different cooperad structure on the cohomology algebras of the open moduli space, in Getzler (1995). We shall only be concerned with the nonsymmetric data arising in the construction, and we may then paraphrase Getzler's construction in the following way.

Define

$$\text{Res}_{\{i,j\}} : A_n \rightarrow (A_{n-n'+1} \otimes A_{n'+1})[-1]$$

to be the Poincaré residue of logarithmic forms along the divisor $u_{ij} = 0$. Shift degrees to obtain a degree zero mapping

$$\Delta_{\{i,j\}}^{\mathbf{A}[-1]} : A_n[-1] \rightarrow A_{n-n'+1}[-1] \otimes A_{n'+1}[-1].$$

Lemma 3.3.2. (Getzler (1995)) The maps $\Delta_{\{i,j\}}^{\mathbf{A}[-1]}$ equip the collection $\mathbf{A}[-1]$ with the structure of an antiplanar dg cooperad.

Following Getzler, we shall call this the *gravity cooperad* and denote it $\mathbf{coGrav} = \mathbf{A}[-1]$.

Lemma 3.3.3. The Poincaré residue has the explicit expression $\text{Res}_{\{i,j\}} = \Delta_{\{i,j\}}^{\mathbf{A}} \circ \partial / \partial \alpha_{ij}$.

Proof. Let $A, B \subset \chi_1(n)$ be a pair of completely crossing subsets and define

$$R_{A,B} = \left(\sum_{\{i,j\} \in A} \alpha_{ij} \right) \left(\sum_{\{k,l\} \in B} \alpha_{kl} \right) \in \mathbb{Q}[\alpha_{ij} \mid \{i,j\} \in \chi_1(n)].$$

Recall that \mathbf{A}_n is freely generated by the α_{rs} modulo the relations that $R_{A,B} = 0$ for all pairs of completely crossing subsets. Assume first that $\{i,j\} \notin A \cup B$. Then

$$\frac{\partial}{\partial \alpha_{ij}} R_{A,B} = 0.$$

Assume conversely, without loss of generality, that $\{i,j\} \in A$. Then

$$\frac{\partial}{\partial \alpha_{ij}} R_{A,B} = \sum_{\{k,l\} \in B} \alpha_{kl}.$$

However, we note that all $\{k,l\} \in B$ must then cross $\{i,j\}$; and since $\Delta_{\{i,j\}}^{\mathbf{A}} \alpha_{rs} = 0$ if $\{r,s\}$ crosses $\{i,j\}$, we can conclude that, in all cases,

$$\Delta_{\{i,j\}}^{\mathbf{A}} \frac{\partial}{\partial \alpha_{ij}} R_{A,B} = 0.$$

This proves that the expression is well-defined as a map on \mathbf{A}_n . That it equals the Poincaré residue is then clear since $u_{ij} = 0$ is the equation defining the strata and $\alpha_{ij} = du_{ij}/u_{ij}$. \square

Definition 3.3.4. Define \mathbf{A}_n^δ to be the joint kernel

$$\mathbf{A}_n^\delta = \bigcap_{\{i,j\} \in \chi_1(n)} \text{Ker}(\text{Res}_{\{i,j\}}) \subset \mathbf{A}_n$$

of all the Poincaré residue maps to boundary strata of Brown's moduli space.

Theorem 3.3.5. (Alm and Petersen (2015); Dupont and Vallette (2015)) The restriction $H^*(M_{0,n}^\delta) \rightarrow \mathbf{A}_n$ is an isomorphism onto \mathbf{A}_n^δ .

The above theorem is equivalent to the statement that the mixed Hodge structure on $H^k(M_{0,n}^\delta)$ is pure of weight $2k$. The maps $\Delta_{\{i,j\}}^{\mathbf{A}}$, defined by regularized restriction, make \mathbf{A}^δ a planar subcooperad of \mathbf{A} . It follows from the theorem that \mathbf{A}^δ , with this cooperad structure, is isomorphic to the planar cooperad $H^*(M_0^\delta)$, with its canonical cooperad structure.

3.4. Brown's theorem on periods. Let ζ denote the algebra of multiple zeta values (for short, MZVs), i.e., the subalgebra of the real numbers generated by 1 and all

$$\zeta(k_1, \dots, k_r) = \sum_{0 < n_1 < \dots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}},$$

for k_1, \dots, k_r a sequence of strictly positive integers, with $k_r \geq 2$. The *weight* of a multiple zeta value $\zeta(k_1, \dots, k_r)$ is the number $k_1 + \dots + k_r$.

Recall the embedded associahedra $X_n \subset M_{0,n}^\delta$. Brown has shown the following very remarkable fact:

Lemma 3.4.1. (Brown (2009)) Let $\beta \in \mathbf{A}_n^\delta$ be a top-degree form. Its integral $\int_{X_n} \beta$ is a rational linear combination of multiple zeta values of weight at most $n - 3$. Moreover, every multiple zeta value arises as such an integral.

4. THE DIHEDRAL KZ CONNECTION

Brown gave a novel treatment of the Knizhnik-Zamolodchikov connection, in Brown (2009), which we recall here.

Definition 4.0.2. Define the *dihedral Lie algebra* on $[n]$, denoted \mathfrak{d}_n , to be the Lie algebra generated by variables δ_{ij} , $i, j \in [n]$, modulo the relations that $\delta_{ji} = \delta_{ij}$ for all indices, $\delta_{ij} = 0$ unless $\{i,j\} \in \chi_1(n)$ is a chord, and

$$[\delta_{i-1j} + \delta_{ij-1} - \delta_{i-1j-1} - \delta_{ij}, \delta_{k-1l} + \delta_{kl-1} - \delta_{k-1l-1} - \delta_{kl}] = 0$$

for all quadruples of indices such that $\#\{i,j,k,l\} = 4$.

Definition 4.0.3. The Lie algebra of spherical braids \mathfrak{p}_n is generated by p_{ij} ($1 \leq i < j \leq n$) with relations

$$\sum_{a=1}^n p_{ak} = 0 \quad \forall k,$$

$$[p_{ij}, p_{kl}] = 0$$

for all indices such that $\#\{i, j, k, l\} = 4$.

The Lie algebra of spherical braids is well-known to be isomorphic (over the rational numbers) to the associated graded Lie algebra of the mapping class group $\Gamma_n = \pi_1(M_{0,n})$. Since $M_{0,n}$ is a $K(\pi, 1)$ space, this means that the Chevalley-Eilenberg cohomology $H^*(\mathfrak{p}_n)$ is isomorphic to the cohomology of $M_{0,n}$.

Remark 4.0.4. Brown (2009) proves that the dihedral Lie algebra \mathfrak{d}_n is isomorphic to the Lie algebra \mathfrak{p}_n of spherical braids. In one direction the isomorphism can be given as follows. Define $[n, n-1]$ to be the set $\{1, \dots, n\}$, but with the non-standard total order $\{n < 1 < \dots < n-1\}$. Then

$$p_{ij} = \delta_{i-1,j} + \delta_{i,j-1} - \delta_{i-1,j-1} - \delta_{i,j},$$

with all sums taken with respect to the totally ordered set $[n, n-1]$. The inverse is

$$\delta_{ij} = \sum_{i < r < s \leq j} p_{rs},$$

again understood as summation with indices in $[n, n-1]$. The isomorphism in particular implies that the Chevalley-Eilenberg cohomology $H^*(\mathfrak{d}_n)$ of the dihedral Lie algebra is isomorphic to the (rational) cohomology of $M_{0,n}$.

Let \mathfrak{t}_{n-1} be the usual Lie algebra of infinitesimal braids. It has generators t_{ij} (for $1 \leq i, j \leq n-1$) and relations $[t_{ij}, t_{kl}] = 0$ for all quadruples of distinct indices, $[t_{ij}, t_{ik} + t_{jk}] = 0$ for all triples of distinct indices, and linear relations $t_{ij} = t_{ji}$ and $t_{ii} = 0$. Its cohomology $H^*(\mathfrak{t}_n)$ is isomorphic to the cohomology of the moduli space $\mathbb{C}^n \setminus \text{diagonals}$ of configurations of n distinct points in the plane. The relations $p_{in} = -\sum_{k=1}^{n-1} p_{ik}$ imply that the Lie algebra \mathfrak{p}_n , hence also \mathfrak{d}_n , is isomorphic to the quotient of \mathfrak{t}_{n-1} by the additional relation $2\sum_{1 \leq i < j \leq n} t_{ij} = 0$.

Lemma 4.0.5. The Lie algebras \mathfrak{d}_n naturally form a planar operad in the category of Lie algebras with direct sum as monoidal product.

Proof. Instead of mimicking gluing of trees, we interpret operadic composition graphically as gluing of polygons. Gluing together a polygon with $n-k+1$ sides and a polygon with $k+1$ sides, along specified edges, produces an n -gon. From the gluing arises a function

$$f \sqcup g : \chi_1(n-k+1) \sqcup \chi_1(k+1) \rightarrow \chi_1(n).$$

We used this previously, when we noted that a chord defines a partition of $\chi_1(n)$ and a corresponding cooperadic cocomposition on the algebra of functions on $M_{0,n}^\delta$.

In this way gluings of polygons define obvious candidates for composition maps

$$\circ_j^i : \mathfrak{d}_{m+1} \oplus \mathfrak{d}_{n+1} \rightarrow \mathfrak{d}_{m+n}.$$

These obviously satisfy the necessary associativity equations, if they are well-defined. To argue that they are, take four distinct vertices $i < k < j < l$ on the $(m+1)$ -gon. Let

$$\mu = \circ_j^{n+1} : \mathfrak{d}_{m+1} \oplus \mathfrak{d}_{n+1} \rightarrow \mathfrak{d}_{m+n}$$

be our candidate map, abusing notation. Then $\mu(p_{kl} + 0) = p_{kl+n-1}$ while

$$\mu(p_{ij} + 0) = \delta_{i-1,j+n-1} + \delta_{i,j-1} - \delta_{i-1,j-1} - \delta_{i,j+n-1} = \sum_{s=j}^{j+n-1} p_{is}.$$

This makes it apparent that $[\mu(p_{ij}), \mu(p_{kl})] = 0$, as required. All other cases of relations are either obviously satisfied or are cyclic rotations of this one. \square

It follows that the Chevalley-Eilenberg cochain complexes (with rational coefficients) $C^*(\mathfrak{d})$ form a planar cooperad of dg commutative algebras.

Lemma 4.0.6. (Brown (2009)) The form $\alpha_n = \sum_{\{i,j\} \in \chi_1(n)} \alpha_{ij} \delta_{ij} \in \mathbf{A}_n \otimes \mathfrak{d}_n$ is a (singular) flat connection on $M_{0,n}^\delta$, compatible with regularized restriction to boundary strata, i.e., it defines (for varying n) a morphism

$$\alpha : C^\bullet(\mathfrak{d}) \rightarrow \mathbf{A}$$

of planar cooperads of dg commutative algebras.

We shall refer to α as the *dihedral Knizhnik-Zamolodchikov connection*, abbreviated as the dihedral KZ connection.

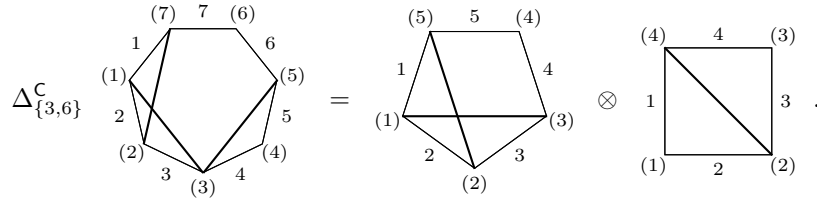
Remark 4.0.7. The dihedral KZ connection is a quasi-isomorphism of planar dg cooperads, because the map is surjective and the two cooperads have the same cohomology, as remarked in 4.0.4. The KZ connection can be regarded in motivic terms as the canonical comparison isomorphism between the Betti and de Rham realizations.

5. DIHEDRAL CHORD DIAGRAMS AND REGULARIZATION

5.1. Dihedral chord diagrams.

Definition 5.1.1. Define \mathbf{C} , the *cooperad of dihedral chord diagrams*, to be the planar cooperad $C^\bullet(\mathfrak{d}/[\mathfrak{d}, \mathfrak{d}])$ of Chevalley-Eilenberg cochain complexes on the Abelianizations of the dihedral Lie algebras.

It follows that $\mathbf{C}_n = \mathbb{Q}[\delta_{ij}^* \mid \{i, j\} \in \chi_1(n)]$. We call elements in this algebra chord diagrams, because their monic monomials can be represented diagrammatically by an n -gon with a set of chords drawn on it. Ordering the chords in the diagram up to an even permutation recovers the monomial, but we shall suppress this detail. A diagrammatic depiction of the cooperadic cocomposition is shown below.

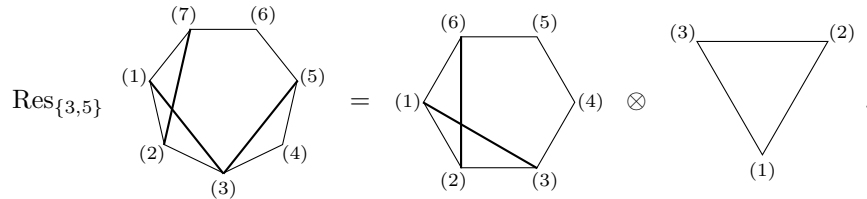


The sides label the operadic inputs and the parenthesized indices at the corners refer to the indices $\{i, j\}$ that label chords.

We can combinatorially mimic the formula 3.3.3 for the Poincaré residue, and set

$$\text{Res}_{\{i,j\}}^{\mathbf{C}} = \Delta_{\{i,j\}}^{\mathbf{C}} \circ \frac{\partial}{\partial \delta_{ij}^*}.$$

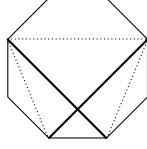
Diagrammatically, this amounts to first removing the chord $\{i, j\}$ from the diagram and then dividing the resulting diagram by cutting the n -gon along $\{i, j\}$. If the chord is not present to begin with, or after its deletion it is not dividing, then the residue is zero. Say that a chord $\{i, j\}$ in a diagram G is *residual* if $\text{Res}_{\{i,j\}}^{\mathbf{C}} G \neq 0$. Below is a diagrammatic example.



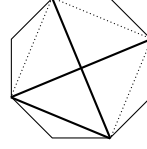
We have here omitted the indices that label the sides. In this example the chord $\{3, 5\}$ is residual for the displayed dihedral chord diagram on the heptagon. It is the only residual chord in the diagram.

Remark 5.1.2. The collection $\mathbf{C}[-1]$ becomes an antiplanar cooperad with the combinatorial Poincaré residues as cocompositions, and the dihedral KZ connection defines a surjection $\mathbf{C}[-1] \rightarrow \mathbf{coGrav}$ onto the antiplanar gravity cooperad.

Assume given a dihedral chord diagram G containing a pair of crossing chords $\{i, j\}$ and $\{k, l\}$. Consider the inscribed quadrilateral formed by the four vertices $\{i, j, k, l\}$. We say that G is *inadmissible* if the side of the quadrilateral that is opposite from the distinguished top side of the polygon is either a side of the polygon or a chord in the diagram. The two forms of inadmissible chord diagrams are illustrated in the figures below; the “inscribed quadrilaterals” mentioned in the definition are depicted by dotted lines. The distinguished top side is drawn with a very thick line.



The opposite side of the quadrilateral is a side of the polygon.



The opposite side of the quadrilateral is a chord in the diagram.

Definition 5.1.3. A dihedral chord diagram is a *gravity chord diagram* if it is not divisible by any inadmissible diagram. Define \mathbf{g}_n to be the set of gravity chord diagrams (identifying two such if they differ only by an ordering of the chords). Define the set of *prime chord diagrams* to be the subset $\mathbf{p}_n \subset \mathbf{g}_n$ of gravity chord diagrams that have no residual chords.

We refer to Alm and Petersen (2015) for a more thorough treatment of gravity chord diagrams.

5.2. Regularization.

Theorem 5.2.1. (Alm and Petersen (2015)) The following assertions are true:

- * The set of forms $\{\alpha_G \mid G \in \mathbf{g}_n\}$ defined by gravity chord diagrams is a basis of \mathbf{A}_n .
- * The set of forms $\{\alpha_P \mid P \in \mathbf{p}_n\}$ defined by prime chord diagrams is a basis of \mathbf{A}_n^δ .
- * The projection $\mathbf{A}[-1] \rightarrow \mathbf{A}^\delta[-1]$ onto prime chord diagrams cogenerates an isomorphism

$$\mathbf{A}[-1] \rightarrow \mathbf{T}^{\text{Pl},-}(\mathbf{A}^\delta[-1])$$

of antiplanar cooperads, i.e., the gravity cooperad $\mathbf{coGrav} = \mathbf{A}[-1]$ is cofree as an antiplanar cooperad, cogenerated by the degree-shifted cohomology algebras of Brown’s moduli spaces.

Definition 5.2.2. We call the projection $\text{reg} : \mathbf{A} \rightarrow \mathbf{A}^\delta$ given by projection onto prime chord diagrams the *regularization*.

Lemma 5.2.3. The regularization is a morphism of planar cooperads.

Proof. Clear, given the diagrammatic interpretation of regularized restriction as cutting along chords. \square

Definition 5.2.4. The composite morphism

$$\alpha_{\text{reg}} : C^\bullet(\mathfrak{d}) \xrightarrow{\alpha} \mathbf{A} \xrightarrow{\text{reg}} \mathbf{A}^\delta$$

of planar dg cooperads is the *regularized dihedral KZ connection*.

Remark 5.2.5. Note that the regularized KZ connection is (unlike the non-regularized version) not a morphism of cooperads of algebras, just a morphism of cooperads.

6. GRAPHS AND RIBBON BRAIDS

Definition 6.0.6. Define \mathbf{rb}_n , the *Lie algebra of spherical ribbon braids*, to be the Lie algebra generated by elements b_{ij} ($1 \leq i, j \leq n$) and s_i ($1 \leq i \leq n$), modulo the relations that $b_{ji} = b_{ij}$, $b_{ii} = 0$, $2s_k + \sum_{i=1}^n b_{ik} = 0$,

$$[b_{ij}, b_{kl}] = 0 \text{ if } \#\{i, j, k, l\} = 4,$$

and $[s_k, \text{anything}] = 0$ for all $k = 1, \dots, n$.

Take $1 < i < n$. We define a map

$$\circ_i^{k+1} : \mathfrak{rb}_{n-k+1} \oplus \mathfrak{rb}_{k+1} \rightarrow \mathfrak{rb}_n$$

by

$$\begin{aligned} b_{pq} \oplus 0 &\mapsto b_{pq} \text{ if } p, q < i, \\ &b_{pq+k-1} \text{ if } p < i, i < q, \\ &\sum_{r=i}^{i+k-1} b_{pr} \text{ if } p < i, q = i, \\ &\sum_{r=i}^{i+k-1} b_{rq+k-1} \text{ if } p = i, i < q, \\ 0 \oplus b_{pq} &\mapsto b_{i+p-1, i+q-1} \text{ if } 1 \leq p, q \leq k, \\ &\sum_{r=1}^{i-1} b_{i+p-1, r} + \sum_{r=i+k}^n b_{i+p-1, r} \text{ if } 1 \leq p \leq k, q = k+1, \\ s_p \oplus 0 &\mapsto s_p \text{ if } p < i, \\ &s_{p+k-1} \text{ if } i < p, \\ &\sum_{i \leq r < s < i+k} b_{rs} + \sum_{i \leq r < i+k} s_r \text{ if } p = i, \\ 0 \oplus s_p &\mapsto s_{i+p-1} \text{ if } 1 \leq p \leq k, \\ &\sum_{r \in [n] \setminus \{i, \dots, i+k-1\}} b_{rs} + \sum_{r \in [n] \setminus \{i, \dots, i+k-1\}} s_r \text{ if } p = k+1. \end{aligned}$$

Lemma 6.0.7. The maps \circ_i^{k+1} defined above make the Lie algebras of spherical ribbon braids a planar operad of Lie algebras.

We shall not prove the above statement. There are similar statements proved in the literature, e.g., the paper Tamarkin (2002) proves that the Lie algebras \mathfrak{t}_n of braids (in the plane) constitute an operad, with analogous formulas for the composition. The paper Ševera (2010) states that the Lie algebras of ribbon braids in the plane form an operad, again by analogous formulas. The operadic composition which we have defined can be given a graphical interpretation, and then the necessary associativity constraints become more transparent.

Define

$$\mathbf{Gra}_n^\circ = \mathbb{Q}[e_{ij}, e_k \mid 1 \leq i < j \leq n-1, 1 \leq k \leq n-1],$$

where we give all generators degree minus one. Note that because of the relations $s_n = -(1/2) \sum_{i=1}^{n-1} b_{in}$ and $b_{ln} = -2s_l - \sum_{i=1}^{n-1} b_{il}$, the mapping

$$\mathbf{Gra}_n^\circ \rightarrow C_*(\mathfrak{rb}_n / [\mathfrak{rb}_n, \mathfrak{rb}_n])$$

that sends e_{ij} to b_{ij} and e_k to s_k , is an isomorphism. We use this isomorphism to transfer the canonical planar operad structure on $C_*(\mathfrak{rb} / [\mathfrak{rb}, \mathfrak{rb}])$ to \mathbf{Gra}° .

Definition 6.0.8. The *operad of tadpole graphs* is the planar dg operad \mathbf{Gra}° .

A monic monomial in \mathbf{Gra}_n° gives rise to a graph with vertex set $\{1, \dots, n-1\}$, an edge between vertices i and j for every e_{ij} in the monomial, and a tadpole (an edge attached at both ends to the same vertex) at the vertex k for every e_k in the monomial, and no legs. By ordering the edges and tadpoles one recovers the monomial from the graph but, just as for diheral chord diagrams we shall mostly suppress this detail. The operad composition

$$\circ_i^{k+1} : \mathbf{Gra}_{n-k+1}^\circ \otimes \mathbf{Gra}_{k+1}^\circ \rightarrow \mathbf{Gra}_n^\circ, \Gamma \otimes \Gamma' \rightarrow \Gamma \circ_i^{k+1} \Gamma'$$

can be described graphically as follows. Remove the vertex i of Γ and consider the edges previously attached to i as legs, producing a graph with legs $\Gamma \setminus \{i\}$. The composition $\Gamma \circ_i^{k+1} \Gamma'$ is the sum $\sum \pm \Gamma''$ over all graphs Γ'' that can be obtained from $\Gamma \setminus \{i\}$ and Γ' by attaching the legs of the former to the vertices of the

and

$$\begin{aligned}
\gamma(-\delta_{42} - \delta_{13}) &= -b_{12} - s_1 - s_2 - b_{23} - s_2 - s_3 \\
&= -b_{12} + \frac{1}{2}(b_{12} + b_{13} + b_{14}) - s_2 - b_{23} + \frac{1}{2}(b_{12} + b_{23} + b_{24}) - s_3 \\
&= \frac{1}{2}b_{13} - \frac{1}{2}b_{23} + \frac{1}{2}(b_{14} + b_{24}) - s_2 - s_3 \\
&= \frac{1}{2}b_{13} - \frac{1}{2}b_{23} - \frac{1}{2}b_{34} - s_4 - s_2 - s_3 \\
&= b_{13} - \frac{1}{2}(b_{13} + b_{23} + b_{34}) - s_2 - s_3 - s_4 \\
&= b_{13} - s_2 - s_4.
\end{aligned}$$

Definition 6.0.12. We abuse notation and denote the induced morphism

$$\gamma : C.(\mathfrak{d}/[\mathfrak{d}, \mathfrak{d}]) \rightarrow C.(\mathfrak{rb}/[\mathfrak{rb}, \mathfrak{rb}]) = \text{Gra}^\circ$$

of planar dg operads by the same symbol γ . Note that s_n or b_{rn} never occur in the formula for $\mathfrak{d}_n \rightarrow \mathfrak{rb}_n$, so above morphism is formally identical to the map on Lie algebras:

$$\gamma(\delta_{ij}) = \sum_{i < r < s \leq j} e_{rs} + \sum_{1 < k \leq j} e_k = e_{i+1} \circ_{i+1}^{j-i+1} 1,$$

the sum taken in the non-standard order $[n, n-1]$.

7. THE MAIN THEOREM

Definition 7.0.13. The *Batalin-Vilkovisky operad* BV is the planar dg operad $H.(\mathfrak{rb})$ of Chevalley-Eilenberg homologies of the spherical ribbon braids.

Remark 7.0.14. The Batalin-Vilkovisky operad is not just planar but, in fact, cyclic. As such it is generated by a bracket operation $\{, \} = [b_{12}] \in H_{-1}(\mathfrak{rb}_3)$, a product operation $m = [1] \in H_0(\mathfrak{rb}_3)$ and a so-called BV operator $\Delta = [s_1] \in H_{-1}(\mathfrak{rb}_2)$, under the following relations:

- * The bracket is a dg Lie bracket of degree -1 .
- * The product operation is a dg commutative associative multiplication.
- * The BV operator Δ squares to zero.
- * The bracket is a degree -1 derivation of the product:

$$\{, \} \circ_2^3 m = m \circ_1^3 \{, \} + (m \circ_n^3 \{, \})(12)$$

- * The BV operator acts as a degree -1 derivation of the bracket:

$$\Delta \circ_1^2 \{, \} + \{, \} \circ_1^2 \Delta + \{, \} \circ_2^2 \Delta = 0.$$

- * The bracket is the obstruction to the BV operator being a derivation of the product:

$$\{, \} = \Delta \circ_1^3 m - m \circ_1^2 \Delta - m \circ_2^2 \Delta.$$

The cyclic structure is defined by the relations $s_n = -(1/2) \sum_{i=1}^{n-1} b_{in}$ and $b_{ln} = -2s_l - \sum_{j=1}^{n-1} b_{jl}$. In particular, if τ is the generator $i \mapsto i+1$ of the cyclic action, then $m\tau = m$ while $\{, \}\tau = -2m \circ_2^2 \Delta - \{, \}$ and $(m \circ_2^2 \Delta)\tau = m \circ_1^2 \Delta + m \circ_2^2 \Delta + \{, \}$.

That $H.(\mathfrak{rb})$ indeed is given by above generators and relations is argued in Ševera (2010).

Theorem 7.0.15. The mapping γ , the regularization, and the dihedral KZ connection together define, via integration over the associahedra in Brown's moduli spaces, a morphism

$$\nu : \zeta \otimes \text{As}_\infty \rightarrow \zeta \otimes \text{BV}$$

of planar dg operads. The coefficients of the operation $\nu_n \in \zeta \otimes \text{BV}_n$ are rational sums of multiple zeta values of weight at most $n-3$.

Proof. It follows from 3.2.1 and 3.4.1 that integration of forms on Brown's moduli spaces is a canonical morphism

$$\zeta \otimes \text{As}_\infty \rightarrow \zeta \otimes (\mathbf{A}^\delta)^*,$$

whose coefficients of the n -ary operation are multiple zeta values of weight at most $n - 3$. The regularized dihedral KZ connection $\alpha_{\text{reg}} : C^\bullet(\mathfrak{d}) \rightarrow \mathbf{A}^\delta$ of 5.2.4 is a morphism of planar dg cooperads. The mapping $\gamma : \mathfrak{d} \rightarrow \mathfrak{rb}$ of 6.0.9 defines a morphism of planar dg cooperads $C(\mathfrak{d}) \rightarrow C(\mathfrak{rb})$. By taking cohomology and dualizing we get $\gamma \circ \alpha_{\text{reg}}^* : (\mathbf{A}^\delta)^* \rightarrow H(\mathfrak{d}) \rightarrow H(\mathfrak{rb}) = \text{BV}$. \square

Definition 7.0.16. We christen the cyclic A_∞ structure ν the *exotic structure*.

7.1. Obtaining an explicit formula. To write a thoroughly explicit formula for the exotic structure we have to choose suitable basis elements of \mathbf{A}^δ and $H^\bullet(\mathfrak{rb})$.

7.1.1. Basis of the BV cooperad. Because of the relations $s_n = -(1/2) \sum_{i=1}^{n-1} b_{in}$ and $b_{ln} = -2s_l - \sum_{j=1}^{n-1} b_{jl}$, the cohomology algebra of the Lie algebra of spherical ribbon braids can be written entirely in terms of b_{ij}^* ($1 \leq i, j \leq n - 1$) and s_k^* ($1 \leq k \leq n - 1$).

Definition 7.1.1. Let $\Lambda(n - 1)$ be the free graded commutative algebra

$$\mathbb{Q}[b_{ij}^* \mid 1 \leq i, j \leq n - 1]$$

on the degree 1 generators b_{ij}^* , modulo the *Arnold relations*

$$b_{ij}^* b_{jk}^* + b_{jk}^* b_{ki}^* + b_{ki}^* b_{ji}^* = 0,$$

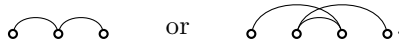
for all triples of distinct indices i, j, k . We call $\Lambda(n - 1)$ the *Arnold algebra*.

Lemma 7.1.2. The cohomology algebra $H^\bullet(\mathfrak{rb}_n)$ is isomorphic to the algebra $\Lambda(n - 1) \otimes \mathbb{Q}[s_1^*, \dots, s_{n-1}^*]$, where we give the generators s_k^* degree one.

The above is rather well-known, see for example Ševera (2010) or Giansiracusa and Salvatore (2012). The proof is a mild generalization of Arnold's computation in Arnol'd (1969), from 1969, of the cohomology of the configuration space of points in the plane in terms of the cohomology of the corresponding braid group.

Proposition 7.1.3. (Alm and Petersen (2015)) Let $B(n - 1)$ be the set of monic monomials in $\Lambda(n - 1)$ such that there are no three indices $1 \leq i < j < k \leq n - 1$ such that the monomial has a factor $b_{ij}^* b_{jk}^*$, and there are no four indices $1 \leq i < j < k < l \leq n - 1$ such that $b_{ik}^* b_{jk}^* b_{jl}^*$ is a factor. Then $B(n - 1)$ is a basis of $\Lambda(n - 1)$.

In the graphical notation of Gra° , identifying monomials with graphs, this means that we exclude graphs containing a subgraph of the form



Define $S(n - 1)^q$ to be the set of degree q monic monomials of $\mathbb{Q}[s_k^*]$, and let $B(n - 1)^p$ be the degree p part of the basis $B(n - 1)$.

Corollary 7.1.4. The set $\beta\sigma_n = \coprod_{p+q=n-3} B(n - 1)^p \sqcup S(n - 1)^q$ is a basis of $H^{n-3}(\mathfrak{rb}_n)$.

7.1.2. Basis of the top dimensional convergent differential forms. Recall from 5.2.1 that we have a basis \mathfrak{p}_n of \mathbf{A}_n^δ in terms of prime chord diagrams.

Definition 7.1.5. Let $\pi_n \subset \mathfrak{p}_n$ be the subset of prime chord diagrams with $n - 3$ chords; this is a basis of the top-degree algebraic forms on $M_{0,n}^\delta$.

Let us be a little bit more specific regarding the combinatorial properties of the elements in π_n .

Define $L(n - 1)$ to be the set of iterated binary bracketings of the indices $1, \dots, n - 1$, subject to the following conditions:

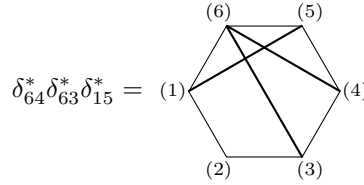
- * Each index appears exactly once. (Thus the word must be an iteration of $n - 2$ binary brackets.)
- * The smallest index in a bracket stands to the left and the largest to the right.

For example, $[1, [2, 3]]$ and $[[1, 2], 3]$ both lie in $L(3)$, but neither $[2, [1, 3]]$ nor $[[1, 3], 2]$ does.

Definition 7.1.6. Say that a binary bracket b (of bracketings) in an $L \in \mathbf{L}(n-1)$ is *connected* if the set of indices appearing inside b is a connected subset of $[n-1]$. Define the set of *prime brackets*, to be denoted $\pi(n-1)$, to be the subset of $\mathbf{L}(n-1)$ consisting of all those P with the property that only the outermost bracket is connected.

Lemma 7.1.7. (Alm and Petersen (2015)) Identifying a bracket enclosing a smallest index i and a largest index j with the chord $\{i-1, j\}$ (subtraction taken cyclically, modulo n), ignoring the outmost bracket, gives an isomorphism $\pi(n-1) \cong \pi_n$ between the set of prime bracketings of $n-1$ indices and the set of prime chord diagrams on the standard n -gon.

For example, the prime bracketing $P = [[[1, 3], 4], [2, 5]]$ has three brackets, ignoring the outmost bracket, namely $[[1, 3], 4]$, $[1, 3]$ and $[2, 5]$. These give the chords $\{6, 4\}$, $\{6, 3\}$ and $\{1, 5\}$, respectively. The corresponding chord diagram on the hexagon is



Thus, the associated form is $\alpha_P = \alpha_{64}\alpha_{63}\alpha_{15} \in \mathbf{A}_6^\delta$. The identification of prime chord diagrams with prime bracketings gives an easy recipe for how to consistently order the chords in a prime diagram. Any order will do, the prime chord diagrams will be a basis whatever convention for how to order the chords we choose, but let us now fix this order so as to make the construction completely unambiguous. Order the brackets by reading them lexicographically outside in and left to right, ignoring the outmost bracket. Thus $[[[1, 3], 4], [2, 5]]$ gives $[[1, 3], 4] < [1, 3] < [2, 5]$. Order the corresponding chords in the same way, viz., $\{6, 4\} < \{6, 3\} < \{1, 5\}$, which we identify with the form $\alpha_{64}\alpha_{63}\alpha_{15}$.

7.1.3. *The explicit formula.* The following result follows from the contents of the preceding two subsections.

Theorem 7.1.8. The n -ary operation of the exotic structure has the following explicit formula:

$$\nu_n = \sum_{P \in \pi_n} \int_{X_n} \alpha_P \otimes g_P \in \zeta \otimes \mathbf{BV}_n.$$

The sum is over the set π_n of prime chord diagrams, defined in 7.1.5. The operation

$$g_P = \sum_{w \in \beta\sigma_n} \langle \alpha_{\text{reg}} \circ \gamma^* w, \alpha_P \rangle \phi_w \in \mathbf{BV}_n$$

is a sum over the basis of $H^{n-3}(\mathfrak{rb}_n)$ mentioned in 7.1.4, with ϕ_w denoting the conjugate dual basis vectors of $H_{3-n}(\mathfrak{rb}_n)$.

Remark 7.1.9. The operation g_P contains as one of its terms the BV operation defined by replacing the outermost bracket in P (considered as a prime bracketing, via 7.1.6) with a product. For example, $g_{[[1,3],[2,4]]}$ contains the operation $\{1, 3\}\{2, 4\}$. This follows immediately from

$$\gamma^*(b_{rs}^*) = \sum_{i < r < s < j} \delta_{ij}^* = \delta_{r-1s}^* + \dots,$$

since the correspondence between prime brackets and prime forms was given by replacing a bracket enclosing r and s with the chord $\{r-1, s\}$. (We are taking the sum with respect to the order $[n, n-1] = \{n < 1 < \dots < n-1\}$.)

7.2. Computation of the first nontrivial term. We shall here follow the algorithmic formula of 7.1.8 to compute the first few terms of the exotic structure. The term ν_3 equals the binary commutative multiplication m . Thus the exotic structure is a deformation of the commutative one. There is no term ν_4 because $A_4^\delta = \mathbb{Q}$. In the case $n = 5$, A_5^δ is one-dimensional in the top degree, spanned by the form

$$\alpha_P = \alpha_{53}\alpha_{14}$$

corresponding to the unique prime bracketing $[[1, 3], [2, 4]]$. To calculate the corresponding integral we fix coordinates s and t on the associahedron X_5 by using the symmetry of the moduli space to gauge the labeled points to

$$z_1, z_2, z_3, z_4, z_5 = 0 < s < t < 1 < \infty.$$

The two relevant dihedral coordinates are then

$$u_{53} = t, \text{ and } u_{14} = 1 - s.$$

Using this, the integral

$$\int_{X_5} \alpha_P = \int_{0 < s < t < 1} \frac{ds dt}{(1-s)t} = \frac{\pi^2}{6} = \zeta(2)$$

equals the second zeta value.

The next step is to calculate the Batalin-Vilkovisky operation g_P . Using the defining relations of A_5 , cf. 3.0.2, we note that

$$\begin{aligned} \alpha_{\text{reg}}(\delta_{53}^* \delta_{14}^*) &= \alpha_P, \quad \alpha_{\text{reg}}(\delta_{52}^* \delta_{14}^*) = -\alpha_P, \quad \alpha_{\text{reg}}(\delta_{52}^* \delta_{13}^*) = \alpha_P, \\ \alpha_{\text{reg}}(\delta_{24}^* \delta_{13}^*) &= -\alpha_P \text{ and } \alpha_{\text{reg}}(\delta_{24}^* \delta_{53}^*) = \alpha_P. \end{aligned}$$

Thus

$$\begin{aligned} g_P &= \sum_{w \in \beta\sigma_5} \langle \alpha_{\text{reg}} \circ \gamma^* w, \alpha_P \rangle \phi_w \\ &= \sum_{w \in \beta\sigma_5} \langle \gamma^* w, \delta_{53}^* \delta_{14}^* - \delta_{52}^* \delta_{14}^* + \delta_{52}^* \delta_{13}^* - \delta_{24}^* \delta_{13}^* + \delta_{24}^* \delta_{53}^* \rangle \phi_w. \\ &= \sum_{w \in \beta\sigma_5} \langle w, \gamma(\delta_{53}\delta_{14}) - \gamma(\delta_{52}\delta_{14}) + \gamma(\delta_{52}\delta_{13}) - \gamma(\delta_{24}\delta_{13}) + \gamma(\delta_{24}\delta_{53}) \rangle \phi_w. \end{aligned}$$

By definition,

$$\begin{aligned} \gamma(\delta_{53}\delta_{14}) &= (b_{12} + b_{13} + b_{23} + s_1 + s_2 + s_3)(b_{23} + b_{24} + b_{34} + s_2 + s_3 + s_4), \\ \gamma(\delta_{52}\delta_{14}) &= (b_{12} + s_1 + s_2)(b_{23} + b_{24} + b_{34} + s_2 + s_3 + s_4), \\ \gamma(\delta_{52}\delta_{13}) &= (b_{12} + s_1 + s_2)(b_{23} + s_2 + s_3), \\ \gamma(\delta_{24}\delta_{13}) &= (b_{34} + s_3 + s_4)(b_{23} + s_2 + s_3), \\ \gamma(\delta_{24}\delta_{53}) &= (b_{34} + s_3 + s_4)(b_{12} + b_{13} + b_{23} + s_1 + s_2 + s_3). \end{aligned}$$

After a little calculation we find that

$$\begin{aligned} &\gamma(\delta_{53}\delta_{14}) - \gamma(\delta_{52}\delta_{14}) + \gamma(\delta_{52}\delta_{13}) - \gamma(\delta_{24}\delta_{13}) + \gamma(\delta_{24}\delta_{53}) \\ &= b_{13}b_{24} + b_{13}b_{23} + b_{12}b_{23} - b_{24}b_{23} + b_{23}b_{34} - b_{12}b_{34} \\ &\quad + b_{13}s_2 - b_{24}s_3 - b_{34}s_3 + b_{12}s_2 + b_{23}s_4 - b_{23}s_1 + b_{34}s_1 - b_{12}s_4 \\ &\quad + s_1s_2 + s_3s_4 - s_1s_4. \end{aligned}$$

Of all these terms, only $b_{12}b_{23}$ and $b_{23}b_{34}$ have zero pairing with all representative cocycles in the preferred basis of $H^2(\mathbf{rb}_5)$. The conjugate operation is

$$\begin{aligned} g_P &= \{1, 3\}\{2, 4\} + \{1, \{2, 3\}\}4 - 1\{\{2, 3\}, 4\} - \{1, 2\}\{3, 4\} \\ &\quad + \{1, 3\}\Delta(2)4 - 1\{2, 4\}\Delta(3) - 12\{\Delta(3), 4\} + \{1, \Delta(2)\}34 \\ &\quad + 1\{2, 3\}\Delta(4) - \Delta(1)\{2, 3\}4 + \Delta(1)2\{3, 4\} - \{1, 2\}3\Delta(4) \\ &\quad + \Delta(1)\Delta(2)34 + 12\Delta(3)\Delta(4) - \Delta(1)23\Delta(4). \end{aligned}$$

The notation we have used here is hopefully self-explanatory (it is, admittedly, not ideal in how it handles signs). The end result of our computations is that

$$\begin{aligned}\nu_5 = & \zeta(2)(\{1, 3\}\{2, 4\} + \{1, \{2, 3\}\}4 - 1\{\{2, 3\}, 4\} - \{1, 2\}\{3, 4\} \\ & + \{1, 3\}\Delta(2)4 - 1\{2, 4\}\Delta(3) - 12\{\Delta(3), 4\} + \{1, \Delta(2)\}34 \\ & + 1\{2, 3\}\Delta(4) - \Delta(1)\{2, 3\}4 + \Delta(1)2\{3, 4\} - \{1, 2\}3\Delta(4) \\ & + \Delta(1)\Delta(2)34 + 12\Delta(3)\Delta(4) - \Delta(1)23\Delta(4)).\end{aligned}$$

7.3. The homotopy in arity 6. We shall here discuss the explicit form of

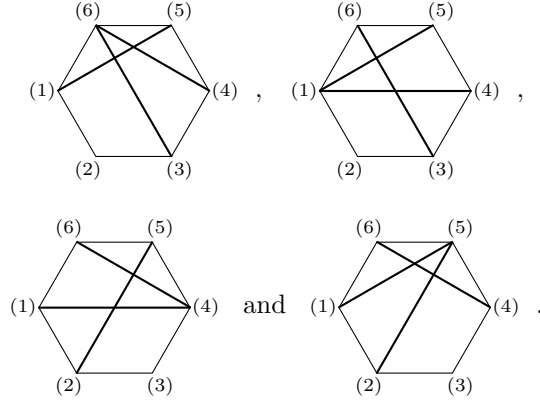
$$\nu_6 = \sum_{P \in \pi_6} \int_{X_6} \alpha_P \otimes g_P.$$

The set of prime chord diagrams under consideration is fairly small, in bracket notation π_6 consists of the four Lie words

$$\begin{aligned}P_1 &= [[[1, 3], 4], [2, 5]], \quad P_2 = [[1, 3], [[2, 4], 5]], \\ P_3 &= [[1, [2, 4]], [3, 5]], \quad \text{and} \quad P_4 = [[1, 4], [2, [3, 5]]],\end{aligned}$$

so we only have to calculate four integrals to find the coefficients. However, the number of terms contributing to the respective operations g_P is huge. We shall accordingly satisfy ourselves with writing out the obvious contributions.

The dihedral chord diagrams corresponding to the four prime bracketings are



It is transparent from this that the associated differential forms are related via the cyclic action of $\tau = (12 \dots 6)$ and, also, via the dihedral flip σ that puts the 6 indices in the order $5 < 4 < 3 < 2 < 1 < 6$. We calculate

$$\begin{aligned}\alpha_{P_1} \cdot \tau &= -\alpha_{P_2}, \quad \alpha_{P_3} \cdot \tau = -\alpha_{P_4}, \\ \alpha_{P_1} \cdot \sigma &= \alpha_{P_4}, \quad \alpha_{P_2} \cdot \sigma = \alpha_{P_3}.\end{aligned}$$

The cyclic action τ acts orientation-preserving while the flip σ acts orientation-reversing on the cell X_6 . It follows that

$$\int_{X_6} \alpha_{P_1} = -\int_{X_6} \alpha_{P_2} = \int_{X_6} \alpha_{P_3} = -\int_{X_6} \alpha_{P_4},$$

so we only have to calculate one of the integrals, say that of α_{P_1} . Fix coordinates

$$0 < x < y < z < 1 < \infty = z_1, z_2, z_3, z_4, z_5, z_6$$

on X_6 . Then

$$u_{64} = z, \quad u_{63} = \frac{y}{z}, \quad u_{15} = 1 - x,$$

and

$$\alpha_{P_1} = d \log u_{64} d \log u_{63} d \log u_{15} = \frac{dx dy dz}{(1-x)yz}.$$

The integral over X_6 equals

$$\int_{X_6} \alpha_{P_1} = \zeta(3),$$

cf. the computations at the end of Brown et al. (2010). It follows that

$$\nu_6 = \zeta(3) \left(\{\{1, 3\}, 4\}\{2, 5\} - \{1, 3\}\{\{2, 4\}, 5\} + \{1, \{2, 4\}\}\{3, 5\} - \{1, 4\}\{2, \{3, 5\}\} + \dots \right).$$

Note that the terms indicated by dots above include not just operations involving the BV operator, but also operations like $\{\{1, \{2, 3\}\}, 4\}5$.

7.4. Homotopical nontriviality of the exotic structure. We can argue that ν is homotopy nontrivial based on the explicit expression for ν_5 .

Proposition 7.4.1. The exotic structure is homotopy nontrivial as a deformation of the strict binary multiplication $m : \mathbf{As} \rightarrow \mathbf{BV}$.

Proof. The first higher homotopy of the exotic structure is the operation ν_5 , which contains the term $\{1, 3\}\{2, 4\}$. This term can be identified with the degree 1 cocycle $b_{13}b_{24}$ in the deformation complex

$$\mathrm{Def}(\mathbf{As}_\infty \xrightarrow{m} C_*(\mathbf{rb})).$$

The deformation complex has two differentials, a differential $\partial_m = [m, \cdot]$ defined by the binary product, and the internal Chevalley-Eilenberg differential ∂_C . We have

$$b_{13}b_{24} = \frac{1}{2}\partial_m(b_{13}b_{12} - b_{13}b_{23}),$$

so by the relation $[b_{ij}, b_{ik} + b_{jk}] = 0$,

$$\begin{aligned} b_{13}b_{24} &= (\partial_m + \partial_C) \frac{1}{2}(b_{13}b_{12} - b_{13}b_{23}) - \partial_C \frac{1}{2}(b_{13}b_{12} - b_{13}b_{23}) \\ &= (\partial_m + \partial_C) \frac{1}{2}(b_{13}b_{12} - b_{13}b_{23}) - \frac{1}{2}([b_{13}, b_{12}] + [b_{13}, b_{23}]) \\ &= (\partial_m + \partial_C) \frac{1}{2}(b_{13}b_{12} - b_{13}b_{23}) + [b_{13}, b_{23}]. \end{aligned}$$

In other words, $b_{13}b_{24}$ is cohomologous to $[b_{13}, b_{23}]$. This latter term can visibly not be ∂_m -exact. \square

7.5. Relationship to Grothendieck-Teichmüller theory. Note that we could just as well have normalized our forms to $\alpha_{ij} = du_{ij}/(cu_{ij})$ for c an arbitrary nonzero constant. Then the coefficient in front of ν_5 would have been $\zeta(2)/c^2$, so the value $\zeta(2)$ is in this elementary sense a “gauge freedom” in our construction. Write ζ^+ for the augmentation ideal of the algebra of multiple zeta values. It is conjectured that

$$\zeta^+ / (\zeta^+ \cdot \zeta^+) \cong \mathbf{grt}_1 \oplus \mathbb{Q}\zeta(2).$$

Here \mathbf{grt}_1 is the Lie coalgebra dual to the Grothendieck-Teichmüller Lie algebra. It follows from this that the Grothendieck-Teichmüller group $GRT = GRT_1 \times G_m$ should act freely as an automorphism group of the algebra of multiple zeta values. The factor G_m acts by arbitrarily fixing a nonzero value for $\zeta(2)$. A recent theorem, with independent proofs by Furusho and Willwacher (Willwacher (2010); Furusho (2010)), says that

$$H^1(\mathrm{Def}(\mathbf{Ass}_\infty \xrightarrow{m} \mathbf{Ger})) \cong \mathbf{grt}_1 \oplus \mathbb{Q}\{1, 3\}\{2, 4\}.$$

Here \mathbf{Ger} is the operad of Gerstenhaber algebras. Just by comparing factors one sees that the class of the operation $\{1, 3\}\{2, 3\}$ should correspond to $\zeta(2)$. The exotic structure is an explicit bridge between above result by Willwacher and Furusho, and the conjectured form of $\zeta^+ / (\zeta^+ \cdot \zeta^+)$. Let us be slightly more detailed. The exotic structure is a Maurer-Cartan element

$$\nu' = \nu - m \in \zeta \otimes \mathrm{Def}(\mathbf{Ass}_\infty \xrightarrow{m} \mathbf{BV}).$$

In fact, ν' has coefficients in ζ^+ . The right hand side in the Maurer-Cartan equation

$$\partial_m \nu' = -\frac{1}{2}[\nu', \nu']$$

consequently has coefficients in the ideal $\zeta^+ \cdot \zeta^+$, so the reduction modulo this ideal is a degree 1 cocycle

$$\overline{\nu}' \in (\zeta^+ / \zeta^+ \cdot \zeta^+) \otimes \text{Def}(\text{Ass}_\infty \xrightarrow{m} \text{BV}).$$

It can equivalently be regarded as a morphism of complexes

$$\text{Def}(\text{Ass}_\infty \xrightarrow{m} \text{BV})'[-1] \rightarrow \zeta^+ / \zeta^+ \cdot \zeta^+$$

from the finite graded dual of the deformation complex. Taking cohomology gives a morphism

$$\varphi_\nu : H^1(\text{Def}(\text{Ass}_\infty \xrightarrow{m} \text{BV}))' \rightarrow \zeta^+ / \zeta^+ \cdot \zeta^+.$$

One can argue that the (graded dual of the) Grothendieck-Teichmüller Lie algebra sits inside the domain of this morphism.

Kontsevich introduced a dg operad **Graphs** of graphs with white and black vertices (Kontsevich (1999)), equipped with a quasi-isomorphism

$$\text{Ger} \rightarrow \text{Graphs}$$

from the operad of Gerstenhaber algebras. It follows from the results by Willwacher and Furusho that

$$H^1(\text{Def}(\text{Ass}_\infty \xrightarrow{m} \text{Graphs})) \cong \mathfrak{grt}_1 \oplus \mathbb{Q} \circ \circ \circ \circ.$$

The correspondence φ_ν , discussed above, relates the class $\circ \circ \circ \circ$ to the coefficient $\zeta(2)$ of

$$\nu_5 = \zeta(2)(\{1, 3\}\{2, 4\} + \dots).$$

The so-called tetraheder element of \mathfrak{grt}_1 has a representative cocycle

$$\Gamma = \circ \circ \circ \circ \circ - \circ \circ \circ \circ \circ + \circ \circ \circ \circ \circ - \circ \circ \circ \circ \circ$$

in the deformation complex. This sum of four graphs corresponds to the displayed terms in our formula

$$\nu_6 = \zeta(3) \left(\{\{1, 3\}, 4\}\{2, 5\} - \{1, 3\}\{\{2, 4\}, 5\} + \{1, \{2, 4\}\}\{3, 5\} - \{1, 4\}\{2, \{3, 5\}\} + \dots \right).$$

It follows that the map φ_ν relates the tetraheder element to $\zeta(3)$.

8. ODD SYMPLECTIC MANIFOLDS

One of the most important classes of Batalin-Vilkovisky algebras are the algebras of functions on odd symplectic manifolds. We show in this section that the exotic deformation can in this case be given an alternative but equivalent formula, more in the spirit of a perturbative quantum field theory and resembling a sum over Feynman diagrams.

Throughout this section we liberally apply the Einstein sum convention of summing over repeated tensor-indices.

Fix a finite number $d > 1$. Assume given d graded variables q^μ ($1 \leq \mu \leq d$) and, additionally, also d graded variables p_ν of degrees $|p_\nu| = 1 - |q^\nu|$. Define the free graded commutative algebra

$$\mathcal{O} = \mathbb{Q}[q^\mu, p_\nu \mid 1 \leq \mu, \nu \leq d].$$

We will be considering this as an algebra of functions on a space. Accordingly we will use geometric terminology and, e.g., refer to elements of the module of derivations

$$\mathcal{X} = \text{Der}(\mathcal{O}, \mathcal{O}) = \mathcal{O} \otimes \text{span}\{\partial_{q^\mu}, \partial_{p_\nu}\}$$

as vector fields. The module of vector fields is a graded Lie algebra. Elements of the dual \mathcal{O} -module,

$$\Omega^1 = \mathcal{O} \otimes \text{span}\{dq^\mu, dp_\nu\},$$

will be called differential one-forms, and the complex

$$\Omega^\bullet = S_\bullet^*(\Omega^1[-1]),$$

with de Rham-type differential d , will be referred to as the de Rham complex and its elements as differential forms. As in differential geometry, we may talk about the contraction $X \lrcorner \beta$ of a vector field X and a differential form β , and define a Lie derivative by $L_X \beta = X \lrcorner d\beta + d(X \lrcorner \beta)$. Our space has a distinguished 2-form, $\omega = dp^\mu dq_\mu$, called the symplectic form. Its d th power ω^d will be called the volume form.

The odd symplectic structure equips \mathcal{O} with a degree -1 Poisson bracket $\{, \}$ (so the suspension $\mathcal{O}[1]$ is a graded Lie algebra, and the adjoint action of bracket acts by graded derivations of the graded commutative product) and a Batalin-Vilkovisky operator

$$\Delta = \frac{\partial^2}{\partial p_\mu q^\mu} : \mathcal{O} \rightarrow \mathcal{O}.$$

The bracket and the Batalin-Vilkovisky operator are related by the formula

$$\{f, g\} = \Delta(fg) - \Delta(f)g - (-1)^{|f|} f \Delta(g).$$

Define

$$D^n(\mathcal{O}, \mathcal{O}) = \text{Map}(\mathcal{O}^{\otimes n}, \mathcal{O}).$$

Then set

$$CD^n(\mathcal{O}, \mathcal{O}) = (D^{n+1}(\mathcal{O}, \mathcal{O}) \otimes_{\mathcal{O}} \Omega^{2d})_X.$$

The coinvariants are taken with respect to the action

$$D \otimes \mu \mapsto X \circ D \otimes \mu + (-1)^{|X||D|} D \otimes L_X \mu$$

of the Lie algebra of vector fields. Observe that $CD^n(\mathcal{O}, \mathcal{O})$ has an action of the permutation group Σ_{n+1} , in particular, it has an action of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$, induced by the natural action on $D^{n+1}(\mathcal{O}, \mathcal{O})$.

Given $D \in D^n(\mathcal{O}, \mathcal{O})$, let $id \cup D$ be the element of $D^{n+1}(\mathcal{O}, \mathcal{O})$ defined by $D \cup id = m \circ (D \otimes id)$, for $m : \mathcal{O} \otimes \mathcal{O} \rightarrow \mathcal{O}$ the multiplication.

Lemma 8.0.1. The map

$$D^n(\mathcal{O}, \mathcal{O}) \xrightarrow{\cup id} D^{n+1}(\mathcal{O}, \mathcal{O}) \xrightarrow{\otimes \omega^d} D^{n+1}(\mathcal{O}, \mathcal{O}) \otimes_{\mathcal{O}} \Omega^{2d} \rightarrow CD^n(\mathcal{O}, \mathcal{O})$$

is a linear isomorphism for all n .

Proof. The lemma is not new, see for example Willwacher and Calaque (2008), but we give a rather complete proof anyway. Take $D \in D^{n+1}(\mathcal{O}, \mathcal{O})$. We can write it as a polydifferential operator. Assume that it is a differential operator of degree at least one in the last input. It is then a sum of operators of the form

$$F^{I_1, \dots, I_n, J} \partial_{I_1} f_1 \dots \partial_{I_n} f_n \partial_a \partial_J g,$$

where the I_k 's and J are multiindices and ∂_a is either ∂_{q^μ} or ∂_{p_ν} . Write

$$F^{I_1, \dots, I_n, J} \partial_{I_1} f_1 \dots \partial_{I_n} f_n \partial_a \partial_J g = F^J \partial_a \partial_J g.$$

By the Leibniz rule we have

$$F^J \partial_a \partial_J g = \pm (\partial_a (F^J \partial_J) - \partial_a F^J \partial_J g)$$

The coordinate vector field ∂_a is Hamiltonian, hence divergence-free (with respect to ω^d), so above equation says that

$$\begin{aligned} & [F^{I_1, \dots, I_n, J} \partial_{I_1} \dots \partial_{I_n} \partial_a \partial_J \otimes \omega^d] \\ &= \pm [\partial_a F^{I_1, \dots, I_n, J} \partial_{I_1} \dots \partial_{I_n} \partial_J \otimes \omega^d] + \sum_{k=1}^n \pm [F^{I_1, \dots, I_n, J} \partial_{I_1} \dots \partial_a \partial_{I_k} \dots \partial_{I_n} \partial_J \otimes \omega^d] \end{aligned}$$

in the quotient space $CD^n(\mathcal{O}, \mathcal{O})$. The terms on the right hand side are operators of order one less in the last input. By induction it follows that every equivalence class $[D\omega^d] \in CD^n(\mathcal{O}, \mathcal{O})$ has a representative with $D = D' \cup id$. Thus the mapping is surjective. Injectivity follows from noting that if

$$L_X(Fg\omega^d) = (FX(g) + \text{div}(FX)g)\omega^d$$

contains no derivative with respect to g , then the vector field FX must be zero. \square

Corollary 8.0.2. The nonsymmetric endomorphism operad of \mathcal{O} is a planar dg operad, i.e., it has a compatible action by cyclic groups.

Proof. The spaces $D^{n-1}(\mathcal{O}, \mathcal{O})$ are the components $\text{End}\langle\mathcal{O}\rangle(n-1)$ of the nonsymmetric endomorphism operad. The previous lemma says that $D^{n-1}(\mathcal{O}, \mathcal{O})$ is linearly isomorphic to the $\mathbb{Z}/n\mathbb{Z}$ -module $CD^{n-1}(\mathcal{O}, \mathcal{O})$; thus, if we define

$$\text{End}\langle\mathcal{O}\rangle_n = \text{End}\langle\mathcal{O}\rangle(n-1),$$

then $\{\text{End}\langle\mathcal{O}\rangle_n\}_{n \geq 3}$ is a planar collection. That the cyclic action is compatible with the nonsymmetric compositions follows from noting that instead of embedding $D^{n-1}(\mathcal{O}, \mathcal{O})$ in $CD^{n-1}(\mathcal{O}, \mathcal{O})$ as the equivalence classes of polydifferential operators of order zero in the n th input, we could just as well have chosen to embed it as the equivalence classes of the operators that have order zero in the k th input, for any $1 \leq k \leq n-1$. \square

Assume given a $2d$ -dimensional odd symplectic manifold (M, ω) . Write \mathcal{A} for the sheaf of Batalin-Vilkovisky algebras of functions on M and take $\phi : \mathcal{A}^{\otimes n-1} \rightarrow \mathcal{A}$ to be a section of the sheaf $\text{Map}(\mathcal{A}^{\otimes n-1}, \mathcal{A})$. Define $\phi\tau$ to be the operator such that

$$\int_M \phi(f_f, f_1, \dots, f_{n-2}) f_{n-1} \omega^d = \pm \int_M \phi\tau(f_1, \dots, f_{n-1}) f_n \omega^d$$

for all functions f_1, \dots, f_n with compact support. The cyclic structure we have defined on the endomorphism operad of \mathcal{O} coincides with the above action $\phi \mapsto \phi\tau$. Above action is maybe more intuitively transparent, but also much more technically involved to define rigorously, since it involves the subtleties of Berezin integration, questions of continuity, etc. The cyclic compatibility of our construction will thus, in the present case, mean that Berezin integration is a trace functional on the exotic structure on \mathcal{A} .

8.1. A cyclic representation. Given $1 \leq i < j \leq n-1$, define

$$D_{ij} = \frac{\partial}{\partial p_\mu^i} \frac{\partial}{\partial q_j^\mu} + \frac{\partial}{\partial p_\mu^j} \frac{\partial}{\partial q_i^\mu} : \mathcal{O}^{\otimes n-1} \rightarrow \mathcal{O}^{\otimes n-1}.$$

Here $\partial/\partial x_i^a = id^{\otimes i-1} \otimes (\partial/\partial x^a) \otimes id^{\otimes n-1-i}$ acts as derivation on the i th factor and leaves the remaining factors untouched. Define also for $1 \leq k \leq n-1$,

$$D_k = \frac{\partial^2}{\partial p_\mu^k \partial q_k^\mu} : \mathcal{O}^{\otimes n-1} \rightarrow \mathcal{O}^{\otimes n-1}. \quad (\text{No sum over } k.)$$

Take a monic monomial $\Gamma \in \text{Gra}_n^\circ$. For every edge e in the graph, write $D_e = D_{ij}$ if $e = e_{ij}$, or $D_e = D_k$ if the edge is a tadpole e_k . Finally define

$$D_\Gamma = m \circ \bigcirc_{e \in \Gamma} D_e : \mathcal{O}^{\otimes n-1} \rightarrow \mathcal{O}.$$

Here $m : \mathcal{O}^{\otimes n-1} \rightarrow \mathcal{O}$ is the commutative product and $\bigcirc_{e \in \Gamma}$ denotes composition. The composition is unambiguous since the monomial Γ defines an ordering of operators D_e up to an even permutation.

Lemma 8.1.1. The association $\Gamma \mapsto D_\Gamma$ is a cyclic representation of Gra° in the endomorphism operad of \mathcal{O} . Moreover, the representation is given by operations that are invariant under affine symplectomorphisms.

Proof. Invariance under translations is clear since the operators D_Γ all have constant coefficients. Invariance under linear symplectomorphisms follows from noting that whenever p_μ appears in a formula it does so in a pair with its conjugate q^μ , with a sum over μ .

That it respects operadic composition repeats the proofs of the various related statements in Kontsevich (1999); Willwacher and Calaque (2008); Willwacher (2010), the grit of which is that the graphical composition rule of removing a vertex and summing over all ways to reconnect edges mimics the Leibniz rule for composition of differential operators. The only part of the lemma not contained in the literature is the cyclic compatibility. It follows from the Leibniz rule, by the following argument. We have

$$\partial_{p_\mu}(f_1 \dots f_{n-1} \partial_{q^\mu} g) = \sum_{i=1}^{n-1} \pm f_1 \dots \partial_{p_\mu} f_i \dots f_{n-1} \partial_{q^\mu} g \pm f_1 \dots f_{n-1} \Delta g.$$

This equation and the analogous one with p and q interchanged, say that

$$[(m \circ (2D_n + \sum_{i=1}^{n-1} D_{in})) \otimes \omega^d] = 0 \in CD^{n-1}(\mathcal{O}, \mathcal{O}),$$

which we can identify as the spherical ribbon braid relation $s_n = -(1/2) \sum_{i=1}^{n-1} b_{in}$. The same argument (with different indexation) shows that also the relations $b_{ln} = -2b_l - \sum_{j=1}^{n-1} b_{jl}$ hold when interpreted in $CD^{n-1}(\mathcal{O}, \mathcal{O})$. These relations define the cyclic group actions on $\text{End}\langle \mathcal{O} \rangle_n = D^{n-1}(\mathcal{O}, \mathcal{O})$ and on Gra_n° . It follows that the representation is, in fact, cyclic. \square

Remark 8.1.2. Instead of using a graphical language, we could have equivalently phrased the preceeding lemma as a cyclic representation $C.(\mathfrak{rb}/[\mathfrak{rb}, \mathfrak{rb}]) \rightarrow \text{End}\langle \mathcal{O} \rangle$.

Remark 8.1.3. The cyclic Batalin-Vilkovisky algebra-structure on the function algebra \mathcal{O} is given by the composite

$$\text{BV} = H.(\mathfrak{rb}) \rightarrow C.(\mathfrak{rb}/[\mathfrak{rb}, \mathfrak{rb}]) \cong \text{Gra}^\circ \xrightarrow{D} \text{End}\langle \mathcal{O} \rangle.$$

8.2. The exotic structure in Darboux coordinates.

Theorem 8.2.1. The operations of the exotic A_∞ structure on $\zeta \otimes \mathcal{O}$ are given by the explicit formula

$$\nu_n = \sum_{\mathbf{c}} \int_{X_n} \alpha_{\text{reg}}(\mathbf{c}) D_{\gamma(\mathbf{c})}, \quad n \geq 3,$$

where the sum is taken over all chord diagrams $\mathbf{c} \in \mathbf{C}_n$ of degree $n - 3$.

Proof. Write coGra° for the cooperad linearly dual to Gra° , and $\text{coBV} = H^*(\mathfrak{rb})$ for the Batalin-Vilkovisky cooperad. By 8.1.3 and 5.2.2, we have a commutative diagram

$$\begin{array}{ccc} \text{coGra}^\circ & \longrightarrow & \mathbf{C} \\ \downarrow & & \downarrow \alpha \\ \text{coBV} & \longrightarrow & \mathbf{A} \xrightarrow{\text{reg}} \mathbf{A}^\delta \end{array}$$

of morphisms of planar dg cooperads. The exotic structure on $\zeta \otimes \mathcal{O}$, given canonically via the composite $\zeta \otimes \text{As}_\infty \rightarrow \zeta \otimes \text{BV} \rightarrow \zeta \otimes \text{coGra}^\circ \rightarrow \zeta \otimes \text{End}\langle \mathcal{O} \rangle$, corresponds to the composite $\text{coGra}^\circ \rightarrow \mathbf{A}^\delta$ via the bottom path in the commutative diagram. The explicit formula in the theorem is the composite $\text{coGra}^\circ \rightarrow \mathbf{A}^\delta$ obtained by following the upper path. \square

The sheaf of functions on an arbitrary odd symplectic $2d$ -dimensional manifold M is locally of the form $C^\infty(T^*[-1]U) = \Gamma(U, \wedge TU)$, where U is an open coordinate chart in \mathbb{R}^d with coordinates q^μ , and p_μ are conjugate sections of the tangent bundle TU such that the symplectic form has the expression $\omega|_U = dp_\mu dq^\mu$. Such coordinates (q, p) are called local *Darboux coordinates* on the odd symplectic manifold. All of the results of this section generalize from the polynomial algebra \mathcal{O} to any Darboux coordinate chart $C^\infty(T^*[-1]U)$. Thus the preceeding theorem can be read as a recipe for how to write the exotic structure on an arbitrary (finite-dimensional) odd symplectic manifold in a Darboux coordinate chart.

8.3. Alternative computation of the first nontrivial term. Let us compute the term ν_5 on \mathcal{O} by using the formula of theorem 8.2.1 of the preceding subsection. Recall from 7.2 that there are five dihedral chord diagrams without residual chords of the right degree:

$$\begin{aligned} \mathbf{c}_i &= \delta_{53}^* \delta_{14}^*, & \mathbf{c}_{ii} &= \delta_{52}^* \delta_{14}^*, & \mathbf{c}_{iii} &= \delta_{52}^* \delta_{13}^*, \\ \mathbf{c}_{iv} &= \delta_{24}^* \delta_{13}^* & \text{and} & & \mathbf{c}_v &= \delta_{24}^* \delta_{53}^*. \end{aligned}$$

By applying $\alpha_{\text{reg}} = \text{reg} \circ \alpha$ we obtain five differential forms, all of them integrating to either plus or minus $\zeta(2)$, just like in 7.2. In the next step we find the five graphs in Gra° associated to the chord diagrams:

$$\begin{aligned} \gamma_i &= (e_{12} + e_{13} + e_{23} + e_1 + e_2 + e_3)(e_{23} + e_{24} + e_{34} + e_2 + e_3 + e_4), \\ \gamma_{ii} &= (e_{12} + e_1 + e_2)(e_{23} + e_{24} + e_{34} + e_2 + e_3 + e_4), \\ \gamma_{iii} &= (e_{12} + e_1 + e_2)(e_{23} + e_2 + e_3), \\ \gamma_{iv} &= (e_{34} + e_3 + e_4)(e_{23} + e_2 + e_3), \\ \gamma_v &= (e_{34} + e_3 + e_4)(e_{12} + e_{13} + e_{23} + e_1 + e_2 + e_3). \end{aligned}$$

Define $\gamma_5 = \gamma_i - \gamma_{ii} + \gamma_{iii} + \gamma_{iv} - \gamma_v$. After some algebra we find that

$$\begin{aligned}\gamma_5 = & e_{13}e_{24} + e_{13}e_{23} + e_{12}e_{23} + e_{23}e_{24} + e_{23}e_{34} - e_{12}e_{34} \\ & + e_{13}e_2 - e_{24}e_3 - e_{34}e_3 + e_{12}e_2 + e_{23}e_4 - e_{23}e_1 + e_{34}e_1 - e_{12}e_4 \\ & + e_1e_2 + e_3e_4 - e_1e_4.\end{aligned}$$

The corresponding operator D_5 in $D^4(\mathcal{O}, \mathcal{O})$ is

$$\begin{aligned}D_5 = & \{1, 3\}\{2, 4\} + \{1, \{2, 3\}\}4 - 1\{\{2, 3\}, 4\} - \{1, 2\}\{3, 4\} \\ & + \{1, 3\}\Delta(2)4 - 1\{2, 4\}\Delta(3) - 12\{\Delta(3), 4\} + \{1, \Delta(2)\}34 \\ & + 1\{2, 3\}\Delta(4) - \Delta(1)\{2, 3\}4 + \Delta(1)2\{3, 4\} - \{1, 2\}3\Delta(4) \\ & + \Delta(1)\Delta(2)34 + 12\Delta(3)\Delta(4) - \Delta(1)23\Delta(4).\end{aligned}$$

To arrive at the above formula we have for example used that

$$D(e_{13}e_{23} + e_{12}e_{23}) = D(e_{12} \circ_2 e_{12}) = \{1, 2\} \circ_2 \{1, 2\}.$$

We can sum up our discussion by

$$\begin{aligned}\nu_5 = & \zeta(2)(\{1, 3\}\{2, 4\} + \{1, \{2, 3\}\}4 - 1\{\{2, 3\}, 4\} - \{1, 2\}\{3, 4\} \\ & + \{1, 3\}\Delta(2)4 - 1\{2, 4\}\Delta(3) - 12\{\Delta(3), 4\} + \{1, \Delta(2)\}34 \\ & + 1\{2, 3\}\Delta(4) - \Delta(1)\{2, 3\}4 + \Delta(1)2\{3, 4\} - \{1, 2\}3\Delta(4) \\ & + \Delta(1)\Delta(2)34 + 12\Delta(3)\Delta(4) - \Delta(1)23\Delta(4)).\end{aligned}$$

This agrees with the earlier computation in 7.2.

9. COMPATIBILITY WITH THE POISSON BRACKET

This section is devoted to proving that the adjoint action of the odd Poisson bracket on a Batalin-Vilkovisky algebra acts by (strict) derivations of the exotic A_∞ structure.

9.1. The moduli space of Riemann spheres with marked points and phase parameters. Let $\text{Conf}_n(Z)$ denote the manifold of all injections of the set $[n]$ into a manifold Z , that is,

$$\text{Conf}_n(Z) = Z^n \setminus \text{diagonals}.$$

Define $\pi : ST(\mathbb{CP}^1) \rightarrow \mathbb{CP}^1$ to be the circle bundle of tangent directions on the Riemann sphere and let V_n be the manifold

$$V_n = \{\xi \in \text{Conf}_n(ST(\mathbb{CP}^1)) \mid \pi \circ \xi \in \text{Conf}_n(\mathbb{CP}^1)\}.$$

Definition 9.1.1. *The open moduli space of Riemann spheres with marked points and phase parameters is the quotient manifold*

$$fM_{0,n} = V_n / PGL_n(\mathbb{C}).$$

The manifold $fM_{0,n}$ has a natural compactification into a manifold with corners, $f\underline{M}_{0,n}$, which can be described as follows, cf. Giansiracusa and Salvatore (2012). Let $\underline{M}_{0,n}$ be the real oriented blow-up of the boundary locus of the Deligne-Mumford compactification $\overline{M}_{0,n}$. Let $L_k \rightarrow M_{0,n}$ be the line bundle whose fiber above a point $[\Sigma]$ is the tangent space $T_{z_k}\Sigma$ at the k th marked point. It extends to a line bundle on $\underline{M}_{0,n}$ and then the product

$$f\underline{M}_{0,n} = SL_1 \times_{\underline{M}_{0,n}} \cdots \times_{\underline{M}_{0,n}} SL_n$$

of the associated circle bundles of directions can be taken as a compactification of $fM_{0,n}$.

Lemma 9.1.2. (Giansiracusa and Salvatore (2012)) The collection $f\underline{M}_0 = \{f\underline{M}_{0,n} \mid n \geq 3\}$ is a cyclic (pseudo-)operad of manifolds with corners. The composition is by gluing surfaces at marked points, tensoring the tangential directions (to produce a nodal surface with a tangential direction at the node).

Let $\pi : fM_{0,n} \rightarrow M_{0,n}$ be the projection that forgets the tangent directions.

Lemma 9.1.3. The cohomology algebra of $f\mathcal{M}_{0,n}$ is the algebra $f\mathcal{A}_n$ generated by the forms $\beta_{ij} = \pi^* \alpha_{ij}$ (for $\{i, j\} \in \chi_1(n)$) and the normalized volume forms θ_k ($1 \leq k \leq n$) on the circle factors of the fibers of $\pi : f\mathcal{M}_{0,n} \rightarrow \mathcal{M}_{0,n}$, modulo the relations that

$$\left(\sum_{\{i,j\} \in A} \beta_{ij} \right) \left(\sum_{\{k,l\} \in B} \beta_{kl} \right) = 0$$

for all pairs of completely crossing subsets $A, B \subset \chi_1(n)$. In short, we have an algebra isomorphism $f\mathcal{A}_n \cong \mathcal{A}_n \otimes \mathbb{Q}[\theta_k \mid 1 \leq k \leq n]$.

Proof. The compactified moduli space $f\mathcal{M}_{0,n}$ is homotopy equivalent to its interior, $f\mathcal{M}_{0,n}$. The latter space is in turn homotopy equivalent to the total space of the $(\mathbb{C}^\times)^n$ -bundle

$$U_{0,n} = L_1^\times \times_{M_{0,n}} \cdots \times_{M_{0,n}} L_n^\times,$$

parametrizing Riemann spheres with nonzero tangent vectors at marked points. Picking a trivialization of the bundle allows one to embed $U_{0,n}$ as the complement of a divisor in $\overline{M}_{0,n} \times \mathbb{C}^n$, and one can deduce that $H^*(U_{0,n})$ is isomorphic to the algebra of logarithmic forms generated by the α_{ij} 's and forms $d \log \tau_k$ ($1 \leq k \leq n$), where τ_k is the coordinate on the k th copy of \mathbb{C} .

The forms θ_k are defined without reference to a trivialization (as they may be considered as fibrewise Haar measures on the principal bundles $SL_k \rightarrow M_{0,n}$) and represent the same cohomology classes as the forms $d \log \tau_k$. \square

Definition 9.1.4. Define $f\mathcal{A}_n^\delta$ to be the image of $\mathcal{A}_n^\delta \otimes \mathbb{Q}[\theta_k \mid 1 \leq k \leq n]$ under the isomorphism $f\mathcal{A}_n \cong \mathcal{A}_n \otimes \mathbb{Q}[\theta_k \mid 1 \leq k \leq n]$.

9.2. Knizhnik-Zamolodchikov with phases. Consider the 1-form

$$\beta = \sum_{\{i,j\} \in \chi_1(n)} \beta_{ij} \gamma(\delta_{ij}) + \sum_{k=1}^n \theta_k s_k$$

with values in the Lie algebra of spherical ribbon braids. Here $\gamma : \mathfrak{d}_n \rightarrow \mathfrak{rb}_n$ is the morphism of the preceding sections. Since $\pi^* : \mathcal{A}_n \rightarrow f\mathcal{A}_n$ is a morphism of algebras, $\beta_{ij} = \pi^* \alpha_{ij}$, γ is a morphism of Lie algebras, and the s_k 's are central, we deduce that β is a flat (singular) connection on $f\mathcal{M}_{0,n}$, i.e., a morphism of dg algebras

$$\beta : C^*(\mathfrak{rb}_n) \rightarrow f\mathcal{A}_n.$$

Note that the regularization map $\text{reg} : \mathcal{A}_n \rightarrow \mathcal{A}_n^\delta$ induces a map $\text{reg} : f\mathcal{A}_n \rightarrow f\mathcal{A}_n^\delta$ which we abusively denote by the same symbol, and define

$$\beta_{\text{reg}} = \text{reg} \circ \beta : C^*(\mathfrak{rb}_n) \rightarrow f\mathcal{A}_n^\delta.$$

We call β_{reg} the *regularized KZ connection with phases*.

9.3. Chains parametrizing homotopy derivations. Recall that we defined $f\mathcal{M}_{0,n} = V_n / PGL_2(\mathbb{C})$, with

$$V_n = \{\xi \in \text{Conf}_n(ST(\mathbb{CP}^1)) \mid \pi \circ \xi \in \text{Conf}_n(\mathbb{CP}^1)\}.$$

Let us first make a remark on an elementary geometric construction. Assume given a point $(z, \sigma) \in ST(\mathbb{CP}^1)$. Flow in the direction σ determines an oriented great circle C through z and its antipodal point $\iota(z)$ on the Riemann sphere. Identify z as the north pole and $\iota(z)$ as the south pole, and let p denote the stereographic projection of $\mathbb{CP}^1 \setminus \{z\}$ onto the plane $\mathbb{C} = T_{\iota(z)}(\mathbb{CP}^1)$ tangent at the antipode. The image $p(C)$ of the great circle is an oriented Euclidean line L in the plane. Let $\tilde{\eta} \in \Gamma(\mathbb{C}, ST(\mathbb{C}))$ be the constant directional vector field defined by the direction of the line L . It lifts to a directional vector field $\eta \in \Gamma(\mathbb{CP}^1 \setminus \{z\}, ST(\mathbb{CP}^1))$ on the Riemann sphere with a zero of index 2 at z .

Definition 9.3.1. Assume $p \geq 0$, $q \geq 2$, $p + q \geq 3$. Define $Y_{p,q} \subset V_{p+q}$ to be the subspace consisting of those $\xi = (z_i, \sigma_i)_{i=1}^{p+q}$ satisfying the following conditions:

- * The direction σ_i at z_i is for all $1 \leq i \leq p + q - 1$ equal to the value of the directional vector field $\eta \in \Gamma(\mathbb{CP}^1 \setminus \{z_{p+q}\}, ST(\mathbb{CP}^1))$ defined by the direction σ_{p+q} at z_{p+q} , i.e., $\sigma_i = \eta|_{z_i}$.

- * If $q \geq 3$, then we require that the images of the points $z_{p+1}, \dots, z_{p+q-1}$ under the stereographic projection $\mathbb{CP}^1 \setminus \{z\} \rightarrow \mathbb{C}$ are collinear on a line parallel to the line L , and ordered compatibly with the orientation of L .

Define $X_{p,q}^o$ to be the image of $Y_{p,q}$ in $fM_{0,p+q}$, and define $X_{p,q}$ to be the closure of $X_{p,q}^o$ in $fM_{0,p+q}$.

It is clear that $X_{p,q}$ is a compact analytic (semialgebraic, in fact) real manifold with corners. Moreover, $X_{0,q}$, for $q \geq 3$, is an associahedron of dimension $q - 3$. To see this, take $\xi = (z_i) \in Y_{0,q}$. We can use the gauge freedom to assume $z_q = \infty$. We are then left with $q - 1$ collinear points $(z_i)_{i=1}^{q-1}$ in the plane, modulo the gauge action $\mathbb{C} \rtimes \mathbb{C}^\times$. (The direction σ_q determines the direction of the line of collinearity.) Use the translation freedom to put z_1 at the origin and the freedom of complex dilation \mathbb{C}^\times to put z_{q-1} at 1. The compatibility between numbering of the points and the orientation of the line ensures we are left with a configuration $0 < z_{p+2} < \dots < z_{p+q-2} < 1$. The compactification $X_{0,q}$ of the open simplex of such configurations is an associahedron: we have an isomorphism $X_{0,q} \cong X_q$ with Brown's associahedron.

The space $X_{1,2}$ is isomorphic to a circle, but not canonically so. We can for example either fix the three points to $z_1, z_2, z_3 = 1, 0, \infty$ and parametrize $X_{1,2}$ by the direction σ_3 , or we can fix the direction σ_3 and parametrize $X_{1,2}$ by the angle t such that $z_1, z_2, z_3 = e^{it}, 0, \infty$.

In general, we can identify $X_{p,q}$ with the following manifold. Define

$$\text{Conf}_{p,q-1}(\mathbb{C}) = \{(z_i)_i \in \text{Conf}_{p+q-1}(\mathbb{C}) \mid \text{Im}(z_{p+1}) = \dots = \text{Im}(z_{p+q-1}), \text{Re}(z_{p+1}) < \dots < \text{Re}(z_{p+q-1})\}$$

to be the subspace consisting of configurations where the last $q - 1$ points are collinear on a line parallel to the real axis and ordered compatibly with the canonical orientation of the line. Putting $z_{p+q} = \infty$ and using the rotation freedom to fix the direction of the line of collinearity defines an isomorphism

$$X_{p,q}^o \cong \text{Conf}_{p,q-1}(\mathbb{C}) / \mathbb{C} \rtimes \mathbb{R}_{>0}.$$

The configuration space $\text{Conf}_{p+q-1}(\mathbb{C})$ has a well-known Axelrod-Singer compactification $\overline{\text{Conf}}_{p,q-1}(\mathbb{C})$ (also called the real Fulton-McPherson compactification). Let $\overline{\text{Conf}}_{p,q-1}(\mathbb{C})$ be the closure inside $\overline{\text{Conf}}_{p+q-1}(\mathbb{C})$. Then above isomorphism extends to

$$X_{p,q} \cong \overline{\text{Conf}}_{p,q-1}(\mathbb{C}) / \mathbb{C} \rtimes \mathbb{R}_{>0}.$$

We can use the translation and dilation freedom to identify $X_{p,q}^o$ with tuples of points $x_j + iy_j$, $1 \leq j \leq p$ and points $0 < t_1 < \dots < t_{q-3} < 1$. The form

$$dx_1 dy_1 \dots dx_p dy_p dt_1 \dots dt_{q-3}$$

then defines an orientation on $X_{p,q}$. This formula is vacuous if $q = 2$, but the only case with $q = 2$ that will play a role in what follows is $X_{1,2}$, which, as discussed above, is isomorphic to a circle. We orient it by θ_3 .

9.4. The compatibility equations. We first note that the forms in the image of $\beta_{\text{reg}} : C^\bullet(\mathfrak{rb}) \rightarrow fA^\delta$ are integrable over the chains $[X_{p,q}]$. Since the points in $X_{0,q}$ correspond to collinear configurations the angle forms θ_k do not contribute to the integrals and

$$\nu_q = \int_{X_{0,q}} \beta_{\text{reg}}$$

will reproduce the exotic structure. Moreover, an integral over $X_{p,q}$ with $p \geq 1$ and $p + q \geq 4$ will vanish because the forms β_{ij} depend holomorphically on the z_1, \dots, z_p . In more detail, we can represent a point in $X_{p,q}^o$ by $q - 2$ points between 0 and 1 on the real axis, and p points that are free to be anywhere in the plane. A top-dimensional form on $X_{p,q}$ must accordingly contain terms $dz_i d\bar{z}_i$, but the KZ connection contains no such anti-holomorphic dependence $d\bar{z}_i$. It follows that the only nonzero integration with $p \geq 1$ is

$$\nu_{1,2} = \int_{X_{1,2}} \beta_{\text{reg}}.$$

Since $A_3^\delta = \mathbb{Q}$, the integrand is

$$\begin{aligned} \theta_1 s_1 + \theta_2 s_2 + \theta_3 s_3 &= -\frac{\theta_1}{2}(b_{12} + b_{13}) - \frac{\theta_2}{2}(b_{12} + b_{23}) - \frac{\theta_3}{2}(b_{13} + b_{23}) \\ &= -\frac{\theta_1 + \theta_2}{2}b_{12} - \frac{\theta_2 + \theta_3}{2}b_{23} - \frac{\theta_1 + \theta_3}{2}b_{13}. \end{aligned}$$

The defining condition $\sigma_i = \eta|_{z_i}$, for $i = 1, 2$ and η the directional vector field determined by σ_3 , implies that if σ_3 rotates with positive orientation, then σ_1 and σ_2 will rotate the same amount in the negative orientation. Since we took θ_3 to define the orientation, we accordingly have

$$\nu_{1,2} = \int_{S^1} \left(-\frac{-\theta_3 - \theta_3}{2} b_{12} - \frac{-\theta_3 + \theta_3}{2} b_{23} - \frac{-\theta_3 + \theta_3}{2} b_{13} \right) = b_{12}.$$

The Batalin-Vilkovisky operation given by b_{12} is the odd Poisson bracket. In the present context however we prefer to regard it as a the adjoint action of “ z_1 ” on “ z_2 ”, with output “ z_3 ”.

Proposition 9.4.1. The adjoint action of the odd Poisson bracket of a Batalin-Vilkovisky algebra acts by strict derivations of the exotic A_∞ structure.

Proof. We will argue that the claim is a consequence of the Stokes’ relation

$$0 = \int_{X_{1,n}} d\beta_{\text{reg}} = \int_{\partial X_{1,n}} \beta_{\text{reg}}.$$

Let us put $X_q = X_{0,q}$ and reserve the notation $X_{p,q}$ for cases when $p \geq 1$. Every codimension one strata of the manifold $X_{1,n}$ has the following form. Assume $S_1 \sqcup S_2 = \{2, \dots, n+1\}$ is a partition, where both subsets are cyclically consecutive, $\#S_1 \geq 1$ and $\#S_2 \geq 2$. Then there is a strata

$$X_{1,S_1 \sqcup \{e_1\}} \times X_{S_2 \sqcup \{e_2\}},$$

corresponding to degenerations where a nodal point develops and the points labeled by $\{1\} \sqcup S_1$ fall on one side of the node and the points labeled by S_2 on the other, and every codimension one strata arises in this way. We can classify the strata further, saying that a strata is of type I if $n+1 \in S_1$, and of type II if $n+1 \in S_2$. Using this, we can write

$$\partial X_{1,n} = \bigcup_{\text{type I}} X_{1,n-q+1} \times X_{q+1} \cup \bigcup_{\text{type II}} X_{n-m+1} \times X_{1,m+1}.$$

Since β_{reg} can produce a top-dimensional form only on spaces X_r and $X_{1,2}$, and since the strata correspond to operad compositions, we find

$$\int_{\partial X_{1,n}} \beta_{\text{reg}} = \pm \nu_{1,2} \circ_2 \nu_n + \sum_{i=2}^{n+1} \pm \nu_{\{2, \dots, n+1\}} \circ_i \nu_{1,2},$$

or, more suggestively,

$$\pm \text{ad}_1(\nu_n(2, \dots, n)) + \sum_i \pm \nu_n(2, \dots, \text{ad}_1(i), \dots, n).$$

Up to verification of the signs this proves that the adjoint action ad_1 of the input labeled 1 is a derivation of the exotic product $\nu_n(2, \dots, n)$ of the remaining inputs. \square

A. CYCLIC COMPATIBILITY OF γ

This appendix completes the proof of 6.0.9 by giving a direct proof of the claim that

$$\gamma : \mathbf{G}_n \rightarrow \mathbf{Gra}_n^\circ, \delta_{ij} \mapsto \sum_{i < r < s \leq j} e_{rs} + \sum_{1 < k \leq j} e_k$$

respects the cyclic group actions. The image of $\gamma(\delta_{r-1, n-1})$ under the cyclic action $\tau : i \mapsto i+1$ is the sum

$$S = e_n + \sum_{r+1 \leq i < j \leq n-1} e_{ij} + \sum_{r+1 \leq i \leq n-1} e_i + \sum_{r+1 \leq i \leq n-1} e_{in},$$

under the proviso that all terms involving an index n are rewritten according to the relations

$$\begin{aligned} e_{an} &= -2e_a - \sum_{1 \leq i \leq n-1} e_{ia}, \\ e_n &= -\frac{1}{2} \sum_{1 \leq j \leq n-1} e_{jn}. \end{aligned}$$

We have to show that $\gamma(\delta_{r-1, n-1}) \cdot \tau = \gamma(\delta_{r, n})$.

It is a computation. Expand the terms containing n and cancel out terms with e_i ,

$$\begin{aligned}
S &= -\frac{1}{2} \sum_{1 \leq i \leq n-1} \left[-2e_i - \sum_{1 \leq j \leq n-1} e_{ji} \right] + \sum_{r+1 \leq i < j \leq n-1} e_{ij} \\
&+ \sum_{r+1 \leq i \leq n-1} e_i + \sum_{r+1 \leq i \leq n-1} \left[-2e_i - \sum_{1 \leq j \leq n-1} e_{ji} \right] \\
&= \sum_{1 \leq i \leq n-1} e_i + \frac{1}{2} \sum_{1 \leq i \leq n-1} \sum_{1 \leq j \leq n-1} e_{ji} + \sum_{r+1 \leq i < j \leq n-1} e_{ij} - \sum_{r+1 \leq i \leq n-1} e_i - \sum_{r+1 \leq i \leq n-1} \sum_{1 \leq j \leq n-1} e_{ji} \\
&= \sum_{1 \leq i \leq r} e_i + \frac{1}{2} \underbrace{\sum_{1 \leq i \leq n-1} \sum_{1 \leq j \leq n-1} e_{ji}}_{=A} + \sum_{r+1 \leq i < j \leq n-1} e_{ij} - \underbrace{\sum_{r+1 \leq i \leq n-1} \sum_{1 \leq j \leq n-1} e_{ji}}_{=B}.
\end{aligned}$$

Now expand the terms A and B . In sums where $j < i$, we switch summation index so that $i < j$.

$$\begin{aligned}
A &= \sum_{1 \leq i \leq n-1} \sum_{1 \leq j \leq n-1} e_{ji} \\
&= \sum_{1 \leq i \leq r} \left(\sum_{1 \leq j < i} e_{ij} + \sum_{i < j \leq n-1} e_{ij} \right) + \sum_{r+1 \leq i \leq n-1} \left(\sum_{1 \leq j < i} e_{ij} + \sum_{i < j \leq n-1} e_{ij} \right) \\
&= \sum_{1 \leq j \leq r} \sum_{1 \leq i < j} e_{ij} + \sum_{1 \leq i \leq r} \left(\sum_{i < j \leq r} e_{ij} + \sum_{r+1 \leq j \leq n-1} e_{ij} \right) + \sum_{r+1 \leq j \leq n-1} \sum_{1 \leq i < j} e_{ij} + \sum_{r+1 \leq i \leq n-1} \sum_{i < j \leq n-1} e_{ij} \\
&= \sum_{1 \leq i < j \leq r} e_{ij} + \sum_{1 \leq i < j \leq r} e_{ij} + \sum_{1 \leq i \leq r} \sum_{r+1 \leq j \leq n-1} e_{ij} \\
&+ \sum_{1 \leq i \leq r} \sum_{r+1 \leq j \leq n-1} e_{ij} + \sum_{r+1 \leq i < j} \sum_{r+1 \leq j \leq n-1} e_{ij} + \sum_{r+1 \leq i \leq n-1} \sum_{i < j \leq n-1} e_{ij} \\
&= 2 \sum_{1 \leq i < j \leq r} e_{ij} + 2 \sum_{1 \leq i \leq r} \sum_{r+1 \leq j \leq n-1} e_{ij} + 2 \sum_{r+1 \leq i \leq n-1} \sum_{i < j \leq n-1} e_{ij}. \\
B &= \sum_{r+1 \leq i \leq n-1} \sum_{1 \leq j < i} e_{ij} + \sum_{r+1 \leq i \leq n-1} \sum_{i \leq j \leq n-1} e_{ij} \\
&= \sum_{r+1 \leq j \leq n-1} \sum_{1 \leq i < j} e_{ij} + \sum_{r+1 \leq i < j \leq n-1} e_{ij} \\
&= \sum_{1 \leq i \leq r} \sum_{r+1 \leq j \leq n-1} e_{ij} + \sum_{r+1 \leq i < j} \sum_{r+1 \leq j \leq n-1} e_{ij} + \sum_{r+1 \leq i < j \leq n-1} e_{ij} \\
&= \sum_{1 \leq i \leq r} \sum_{r+1 \leq j \leq n-1} e_{ij} + \sum_{r+1 \leq i \leq n-1} \sum_{i < j \leq n-1} e_{ij} + \sum_{r+1 \leq i < j \leq n-1} e_{ij}.
\end{aligned}$$

When the expressions for A and B are inserted into the sum S , most terms cancel out.

$$\begin{aligned}
S &= \sum_{1 \leq i \leq r} e_i + \left[\sum_{1 \leq i < j \leq r} e_{ij} + \sum_{1 \leq i \leq r} \sum_{r+1 \leq j \leq n-1} e_{ij} + \sum_{r+1 \leq i \leq n-1} \sum_{i < j \leq n-1} e_{ij} \right] \\
&+ \sum_{r+1 \leq i < j \leq n-1} e_{ij} - \left[\sum_{1 \leq i \leq r} \sum_{r+1 \leq j \leq n-1} e_{ij} + \sum_{r+1 \leq i \leq n-1} \sum_{i < j \leq n-1} e_{ij} + \sum_{r+1 \leq i < j \leq n-1} e_{ij} \right] \\
&= \sum_{1 \leq i \leq r} e_i + \sum_{1 \leq i < j \leq r} e_{ij}.
\end{aligned}$$

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STOCKHOLM UNIVERSITY, DEPARTMENT OF MATHEMATICS
E-mail address: alm@math.su.se